Math 3331 – ODEs

We now solve 2nd order ODEs of the form $ay'' + by' + cy = 0$ a, b, c constants

Consider $y' = y$

$\frac{dy}{y} = dx$ $\ln y = x + \ln c$

so $y = ce^{x}$ suppress c (set $c = 1$)

or $y' = -2y$

$\frac{dy}{y} = -2dx$

$\ln y = -2x \Rightarrow y = e^{-2x}$

in general

$\frac{dy}{dx} = my$ $y = e^{mx}$ look for sol^n of this form so $y' = me^{mx}, \quad y'' = m^2 e^{mx}$

sub $ay'' + by' + cy = 0$

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$(am^2 + bm + c)e^{mx} = 0$$

$e^{mx} \neq 0 \Rightarrow$ $\boxed{am^2 + bm + c = 0 \quad \text{characteristic eq}^n}$

the roots will either be

(i) real distinct

(ii) real repeated

(iii) complex

<u>Ex1</u> $y'' + 3y' - 10y = 0$

CE $m^2 + 3m - 10 = 0$

$$(m+5)(m-2) = 0 \Rightarrow m = -2, 5$$

Two indep. sol^n

$$y_1 = e^{-2x}, \quad y_2 = e^{5x}$$

check Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^{-2x} & e^{5x} \\ -2e^{-2x} & 5e^{5x} \end{vmatrix} = 5 \cdot e^{-2x} e^{5x} + 2e^{-2x} e^{5x}$$

$$= 7e^{3x} \neq 0 \checkmark$$

G.S. $\quad y = c_1 e^{-2x} + c_2 e^{5x}$

$\underline{Ex2}$ Solve $\quad y'' - 3y' + 2y = 0 \quad y(0) = 4$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad y'(0) = 5$

CE $\quad m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1, 2$

$$y = c_1 e^{x} + c_2 e^{2x}$$

$$y' = c_1 e^{x} + 2c_2 e^{2x}$$

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 4 \\ y'(0) &= c_1 + 2c_2 = 5 \end{aligned}\right\} \begin{aligned} c_1 &= 3 \\ c_2 &= 1 \end{aligned}$$

$$y = 3e^{x} + e^{2x}$$

(2) Repeated Roots

Ex 3 $\quad y'' - 6y' + 9y = 0$

$m^2 - 6m + 9 = 0 \quad (m-3)^2 = 0 \quad m = 3, 3$

1 sol^n $\quad y = e^{3x}$

for 2nd sol^n use reduction of order

$$y = e^{3x}u, \quad y' = e^{3x}u' + 3e^{3x}u$$

$$y'' = e^{3x}u'' + 3e^{3x}u' + 3e^{3x}u' + 9e^{3x}u$$

$$\Rightarrow \quad e^{3x}u'' + \cancel{6e^{3x}u'} + \cancel{9e^{3x}u}$$
$$- 6(\cancel{e^{3x}u'} + \cancel{3e^{3x}u})$$
$$+ \cancel{9e^{3x}u} = 0$$

$$u'' = 0 \qquad u' = 1 \qquad u = x$$

$$y_2 = xe^{3x}$$

G.S. $\quad y = c_1 e^{3x} + c_2 x e^{3x}$

In general for repeated roots

$$m = r, r$$

$$y_1 = e^{rx}, \quad y_2 = x e^{rx}$$

Ex 4 $\quad y'' + 2y' + y = 0 \qquad y(0) = 2$

$m^2 + 2m + 1 = 0 \qquad y'(0) = -4$

$m = -1, -1$

G.S. $\quad y = c_1 e^{-x} + c_2 x e^{-x}$

$$y' = -c_1 e^{-x} + c_2 e^{-x} - c_2 x e^{-x}$$

$y(0) = \quad c_1 = 2$

$y'(0) = \quad -c_1 + c_2 = -4 \qquad c_2 = -4 + 2 = 2$

So $\quad y = 2e^{-x} + 2xe^{-x}$