Math 2371 Calc III
Sample Test 3 - Solns

1. Evaluate the following line integrals:

(i) $\int_C xy\,ds$ where C counter clockwise direction along the circle $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$.

Soln. We use the parametrization $x = 2\cos t,\ y = 2\sin t$ where $t = 0 \to \pi/2$.
Now $\dfrac{dx}{dt} = -2\sin t$ and $\dfrac{dy}{dt} = 2\cos t$ giving

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}dt = \sqrt{(-2\sin t)^2 + (2\cos t)^2}dt = 2dt$$

the line integral is

$$\int_0^{\pi/2} 8\sin t\cos t\,dt = 4\sin^2 t\Big|_0^{\pi/2} = 4.$$

(ii) $\int_C (x+y)dx - 2xdy$ where C is the parabola $y = x^2$ from $(0,0)$ to $(2,4)$.

Soln. If $y = x^2$ then $dy = 2xdx$ and the line integral becomes

$$\int_0^2 (x + x^2)dx - 2x\cdot 2xdx = \int_0^2 \left(x - 3x^2\right)dx = \frac{1}{2}x^2 - x^3\Big|_0^2 = -6.$$

2. Is the following vector field conservative?

$$\vec{F} = < 2xy, x^2 + z^2, 2yz > .$$

Soln. Since $\nabla \times \vec{F} = 0$ (you need to show this) then the vector field is conservative. Thus f exists such that $\vec{F} = \vec{\nabla} f$ so

$$\begin{aligned} f_x &= 2xy & &\Rightarrow & f &= x^2y + A(y,z) \\ f_y &= x^2 + z^2 & &\Rightarrow & f &= x^2y + yz^2 + B(x,z) \\ f_z &= 2yz & &\Rightarrow & f &= yz^2 + C(x,y) \end{aligned}$$

Therefore we see that

$$f = x^2y + yz^2 + c$$

(we set $c = 0$ for the next part) and

$$\int_C 2xy\,dx + (x^2 + z^2)\,dy + 2yz\,dz = x^2y + yz^2\Big|_{(1,1,1)}^{(2,3,4)} = 58$$

3. Evaluate the following line integral $\int_C 2y\,dx + x\,dy + dz$ where C is the line joining $(0,1,2)$ to $(1,-2,2)$.

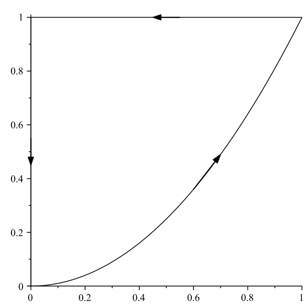
Soln. Here, we need the parametric equation that connects the points. If $P(0,1,2)$ to $Q(1,-2,2)$ are the points, the vector that connects the points is $\vec{PQ} = < 1,-3,0 >$ and the line is $x = 0+t$, $y = 1-3t$ and $z = 2$. To evaluate the line integral we need the differentials. These are $dx = dt$, $dy = -3dt$ and $dz = 0$. Thus the line integral becomes

$$\int_0^1 2(1-3t)\,dt + t(-3)dt = \int_0^1 (2-9t)dt = 2t - \frac{9}{2}t^2\Big|_0^1 = -\frac{5}{2}.$$

4. Green's Theorem is

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Verify Green's Theorem where $\vec{F} = < y^2, x^2 + 2xy >$ where R is the region bound by the curves $y = x^2$, $y = 1$ and $x = 0$ in $Q1$ (see the figure below).



(a) #4.

Soln. Here, we have three separate curves which we denote by C_1, C_2 and C_3.

C_1 : So $y = x^2, dy = 2x\,dx$ so $\int_0^1 x^4 dx + (x^2 + 2x^3)2x\,dx = 3/2$

C_2 : So $y = 1, dy = 0$ so $\int_1^0 dx = -1$

C_3 : So $x = 0, dx = 0$ so $\int_{C_3} 0 = 0$

Thus $\int_c y^2 dx + (x^2 + 2xy)dy = 3/2 - 1 + 0 = 1/2$.

For the second part, $P = y^2$ and $Q = x^2 + 2xy$ then $Q_x - P_y = 2x + 2y - 2y = 2x$. Therefore

$$\iint_R (Q_x - P_y)\,dA = \int_0^1 \int_{x^2}^1 2x dy\,dx = 1/2.$$