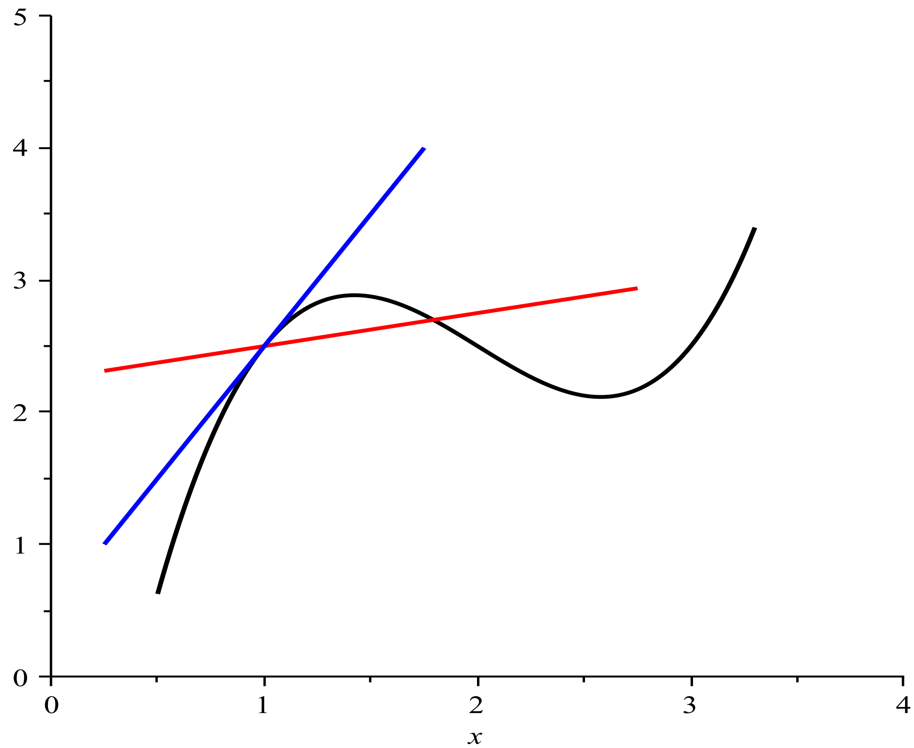
Calculus 3 - Partial Derivatives

In calculus 1 we introduced the derivative. We considered the function $y = f(x)$ and a secant to the curve that goes through the points $(x, f(x))$ and $(x+h, f(x+h))$ (in red). Then we let $h \to 0$ and the secant line (red) becomes the tangent line (blue)



Mathematically, we define the derivative of a function $y = f(x)$ as

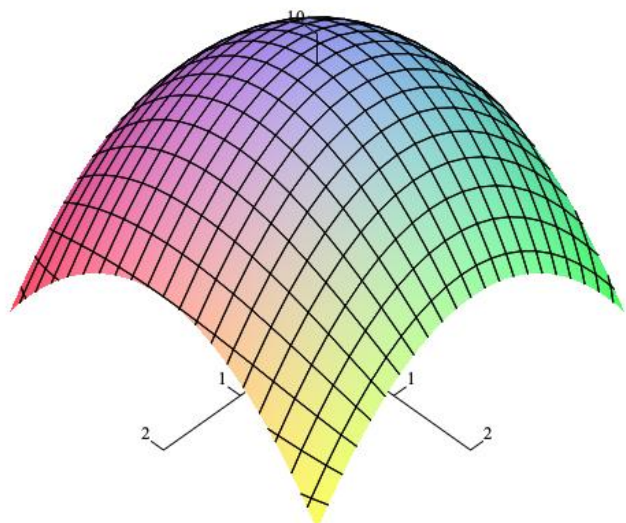
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

So can we define the derivative for functions of more than one independent variable? Here we will consider functions of two independent variables

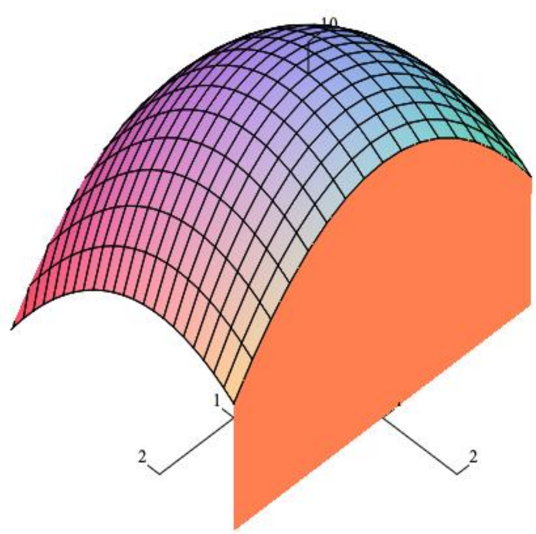
$$z = f(x, y) \tag{2}$$

but these ideas certainly extended to an arbitrary number of independent variables.

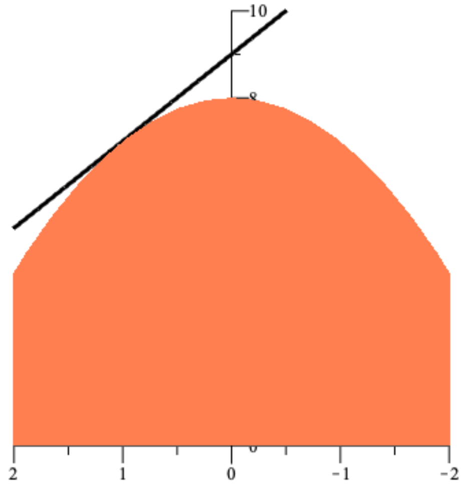
Consider some surface $z = f(x, y)$



Here, we will take a slice where we fix y to some value and vary x



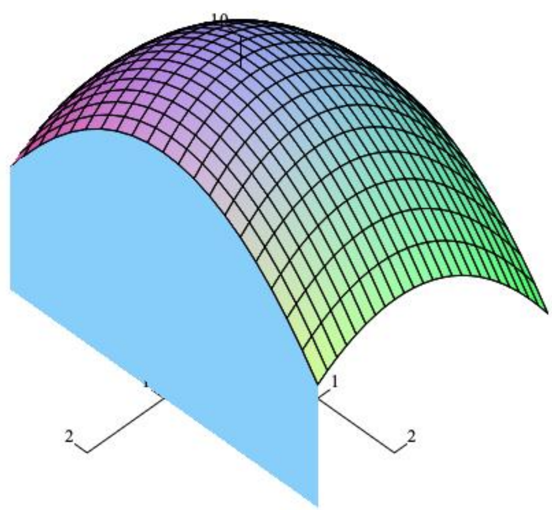
If we look straight down the y axis we see This certain looks like some-

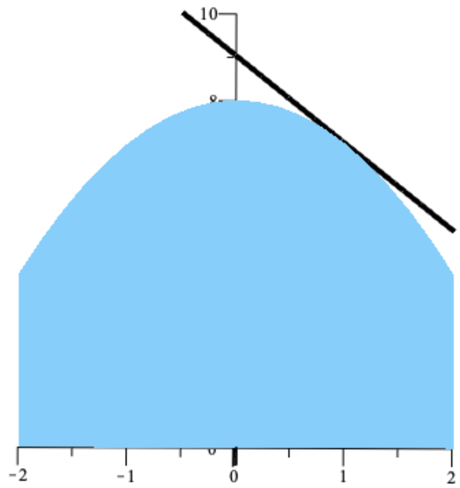


thing from Calc 1 and so we can find a tangent line. So mathematically what we have done is

$$\lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \tag{3}$$

Similarly, we fix x and vary y and look straight down the x axis we see





Again, this looks like something from Calc 1 and so we can find a tangent line. So mathematically what we have done is

$$\lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k} \tag{4}$$

So now we define two different derivatives, an x derivative and the y derivative and are defined as

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \tag{5a}$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}. \tag{5b}$$

as there are two of them we call these *partial derivatives*.

Abbreviations

As we have abbreviations for ordinary derivatives like y' or f' we also have abbreviations for partial derivatives. These would be

$$\frac{\partial f}{\partial x} = f_x = z_x, \quad \frac{\partial f}{\partial y} = f_y = z_y \tag{6}$$

So let's look at an example. Consider

$$f(x,y) = 2x - y \tag{7}$$

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}, \\ &= \lim_{h\to 0} \frac{(2(x+h) - y) - (2x-y)}{h} \\ &= \lim_{h\to 0} \frac{2x + 2h - y - 2x + y}{h} \\ &= \lim_{h\to 0} \frac{2h}{h} \\ &= 2\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}, \\ &= \lim_{k\to 0} \frac{2x - (y+k) - (2x-y)}{h} \\ &= \lim_{k\to 0} \frac{2x - y - k - 2x + y}{h} \\ &= \lim_{k\to 0} \frac{-k}{k} \\ &= -1\end{aligned} \tag{9}$$

so

$$f_x = 2, \quad f_y = -1. \tag{10}$$

One thing to note that the partial derivatives are usually different!

Here's another example

$$f(x,y) = x^2 e^y \tag{11}$$

so

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}, \\
&= \lim_{h\to 0} \frac{(x+h)^2 e^y - x^2 e^y}{h} \\
&= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2 - x^2) e^y}{h} \\
&= \lim_{h\to 0} \frac{(2x+h) h e^y}{h} \\
&= \lim_{h\to 0} (2x+h) e^y \\
&= 2xe^y
\end{aligned} \tag{12}$$

So you're probably thinking - is there a short cut? Well yes. Since we defined these derivatives as fixing one variable and letting the other vary we are essential treating the fixed variable as constant so in the previous example

$$\frac{\partial f}{\partial x} = \frac{\partial \left(x^2 e^y\right)}{\partial x} = \frac{\partial \left(x^2 e^c\right)}{\partial x} = \frac{\partial \left(x^2\right)}{\partial x} e^c = 2xe^c = 2xe^y \tag{13}$$

To calculate the y derivative we would

$$\frac{\partial f}{\partial y} = \frac{\partial \left(x^2 e^y\right)}{\partial y} = \frac{\partial \left(c^2 e^y\right)}{\partial y} = c^2 \frac{\partial \left(e^y\right)}{\partial y} = c^2 e^y = x^2 e^y \tag{14}$$

With some practice, you won't have to always replace the fixed variable with a c.

Example Calculate the partial derivatives for $z = \ln(x^2 + xy^4 + 1)$

Here we use the change rule first so

$$z_x = \frac{1}{x^2 + xy^4 + 1} \cdot (2x + y^4) \tag{15}$$

Similarly

$$z_y = \frac{1}{x^2 + xy^4 + 1} \cdot 4xy^3 \tag{16}$$

More independent Variables

Partial derivatives also apply for functions with more independent variables so, for example, if $T(x, y, z) = ze^{x^2+y^2}$ then

$$T_x = 2xze^{x^2+y^2}, \quad T_y = 2yze^{x^2+y^2}, \quad T_z = e^{x^2+y^2}. \tag{17}$$

Implicit Differentiation

Recall from Calc 1 on differentiation

$$x^2 + y^2 = 1 \tag{18}$$

We certainly could solve for y

$$y = \pm\sqrt{1 - x^2} \tag{19}$$

but instead we calculate

$$2x + 2yy' = 0 \tag{20}$$

and solving for y' gives

$$y' = -\frac{x}{y} \tag{21}$$

Similar for partial derivatives. Consider

$$x^2 + y^2 + z^2 = 1 \tag{22}$$

We could solve for z so

$$z = \pm\sqrt{1 - x^2 - y^2} \tag{23}$$

so

$$z_x = \pm\frac{1}{2}(1 - x^2 - y^2)^{-1/2} \cdot 2x = \pm\frac{x}{\sqrt{1 - x^2 - y^2}}. \tag{24}$$

Instead we will differentiate (22) implicitly so

$$2x + 2zz_x = 0 \tag{25}$$

and solving for z_x gives

$$z_x = -\frac{x}{z} \tag{26}$$

We see that substituting (23) into (26) gives (24)!

Let's consider this in general. If our function is defined as

$$F(x, y, z) = 0 \tag{27}$$

where $z = f(x, y)$ then we differentiate (27) implicitly so

$$F_x + F_z \cdot z_x = 0, \quad F_y + F_z \cdot z_y = 0 \tag{28}$$

and solving for z_x and z_y gives

$$z_x = -\frac{F_x}{F_z}, \quad z_y = -\frac{F_y}{F_z}. \tag{29}$$

As a check consider (22). We define F as

$$F = x^2 + y^2 + z^2 - 1 \tag{30}$$

We calculate

$$F_x = 2x, \quad F_y = 2y, \quad F_z = 2z \tag{31}$$

and from (29) we obtain

$$z_x = -\frac{2x}{2z}, \quad z_y = -\frac{2y}{2z}. \tag{32}$$

Note: Move everything to one side of the equal sign before defining F.

Example Calculate z_x and z_y for $x^2z + y^3 - yz^4 = 7x - y + 2$.

Soln: We first move everything to one side of the equal sign and name this so so

$$F = x^2z + y^3 - yz^4 - 7x + y - 2 \tag{33}$$

Then calculate F_x, F_y and F_z

$$F_x = 2xz - 7, \quad F_y = 3y^2 - z^4 + 1, \quad F_z = x^2 - 4yz^3 \tag{34}$$

Then use the formulate (29) so

$$z_x = -\frac{F_x}{F_z} = -\frac{2xz - 7}{x^2 - 4yz^3}, \quad z_y = -\frac{F_y}{F_z} = -\frac{3y^2 - z^4 + 1}{x^2 - 4yz^3} \tag{35}$$

Higher Order Derivatives

In Calc 1 we calculated $y', y'', y''', y^{(4)}$ etc. We can do the same with partial derivatives.

Consider $f(x, y) = x^5y^3 + (2x + 3y)^4$. We first calculate the first derivatives

$$f_x = 5x^4y^3 + 4(2x + 3y)^3 \cdot 2, \quad f_y = 3x^5y^2 + 4(2x + 3y)^3 \cdot 3 \tag{36}$$

Next we calculate derivatives of derivatives. So how many are there of these well

$$f_{xx}, \quad f_{xy}, \quad f_{yx}, \quad f_{yy} \tag{37}$$

In calculating these we obtain

$$\begin{aligned} f_{xx} &= 20x^3 + 48(2x + 3y)^2 \\ f_{xy} &= 15x^4y^2 + 72(2x + 3y)^2 \\ f_{yx} &= 15x^4y^2 + 72(2x + 3y)^2 \\ f_{yy} &= 6x^5y + 108(2x + 3y)^2 \end{aligned} \tag{38}$$

Notice that $f_{xy} = f_{yx}$. So

$$\begin{gathered} f_x \quad f_y \\ f_{xx} \quad f_{xy} \quad f_{yy} \\ f_{xxx} \quad f_{xxy} \quad f_{xyy} \quad f_{yyy} \\ f_{xxxx} \quad f_{xxxy} \quad f_{xxyy} \quad f_{xyyy} \quad f_{yyyy} \end{gathered}$$