Math 4315 PDE's

so given

$$a u_{xx} + b u_{xy} + c u_{yy} + \text{lots} = 0$$

when $b^2 - 4ac > 0$ (hyperbolic)

we can transform to standard Form (modified)

$$u_{rs} + \text{lots} = 0.$$

why? The regular form was too hard as we need to solve

$$a r_x^2 + b r_x r_y + c r_y^2 = -(a s_x^2 + b s_x s_y + c s_y^2)$$

$$2a r_x s_x + b(r_x s_y + r_y s_x) + 2c r_y s_y = 0$$

and solving

$$a r_x^2 + b r_x r_y + c r_y^2 = 0$$

$$a s_x^2 + b s_x s_y + c s_y^2 = 0$$

was easier.

For example. $u_{xx} - 4x^2 u_{yy} + 2u_y = 0$

note: lower order term u_y

so $b^2 - 4ac = 16x^2 > 0$ (for $x \neq 0$)

$r_x^2 - 4x^2 r_y^2 = 0$ $\quad$ $s_x^2 - 4x^2 s_y^2 = 0$

$(r_x - 2x r_y)(r_x + 2x r_y) = 0$

$r_x - 2x r_y = 0$ $\quad$ $s_x + 2x s_y = 0$

$\frac{dx}{1} = \frac{dy}{-2x}$; $dr = 0$ $\quad$ $\frac{dx}{1} = \frac{dy}{2x}$; $ds = 0$

$r = R(x^2 + y)$ $\quad$ $s = S'(x^2 - y)$

choose $r = x^2 + y$, $s = x^2 - y$ $\quad$ $\left[x^2 = \frac{r+s}{2},\ y = \frac{r-s}{2} \right]$

so $r_x = 2x$, $r_y = 1$, $r_{xx} = 2$ $\quad$ $r_{xy} = r_{yy} = 0$

$s_x = 2x$ $\quad$ $s_y = -1$ $\quad$ $s_{xx} = 2$ $\quad$ $s_{xy} = s_{yy} = 0$

$u_y = u_r r_y + u_s s_y = u_r - u_s$

$u_{xx} = 4x^2 u_{rr} + 8x^2 u_{rs} + 4x^2 u_{ss} + 2u_r + 2u_s$

$u_{yy} = u_{rr} - 2u_{rs} + u_{ss}$

& $u_{xx} - 4x^2 u_{yy} + 2u_y = 0$

$\Rightarrow$ $4x^2 \cancel{u_{rr}} + 8x^2 u_{rs} + 4x^2 \cancel{u_{ss}} + 2u_r + 2\cancel{u_s}$

$- 4x^2 u_{rr} + 8x^2 u_{rs} - 4x^2 u_{ss} + 2u_r - 2u_s = 0$

$16x^2 u_{rs} + 4u = 0$ $\quad$ $u_{rs} + \frac{u_r}{4x^2} = 0$ $\quad$ $u_{rs} + \frac{u_r}{2(r+s)} = 0$

So we ask - can we hit a regular SF?

Consider

$$u_{xx} - u_{yy} = 0$$

To transform to H Hyp

$$r_x^2 - r_y^2 = 0 \quad s_x^2 - s_y^2 = 0$$

$$(r_x - r_y)(r_x + r_y) = 0$$

$$r_x - r_y = 0 \quad s_x + s_y = 0$$

$$\frac{dx}{1} = \frac{dy}{-1}; \; dr = 0 \quad \frac{ds}{1} = \frac{dy}{-1}; \; ds = 0$$

$$r = R(x+y) \quad s = S(x-y)$$

$$r = x+y, \quad s = x-y$$

$$\left.\begin{aligned} u_{xx} &= u_{rr} + 2u_{rs} + u_{ss} \\ u_{yy} &= u_{rr} - 2u_{rs} + u_{ss} \end{aligned}\right\} \quad u_{xx} - u_{yy} = 0 \Rightarrow 4u_{rs} = 0$$

So can we go from $u_{rs} = 0 \Rightarrow u_{xx} - u_{yy} = 0$

So $x = \frac{r+s}{2}, \quad y = \frac{r-s}{2}$

$$
\begin{tikzcd}
\text{PDE} \arrow[r] \arrow[rr, bend right] & u_{rs} + \cdots = 0 \arrow[r] & u_{\bar r\bar r} - u_{\bar s\bar s} + \cdots = 0
\end{tikzcd}
$$

Can we go directly?

so $\bar r = \frac{r+s}{2}$, $\bar s = \frac{r-s}{2}$

Previous example

$$u_{xx} - 4x^2 u_{yy} + 2u_y = 0$$

$$r = x^2 + y \quad s = x^2 - y$$

$$\bar r = \frac{r+s}{2} = \frac{2x^2}{2} = x^2, \quad \bar s = \frac{r-s}{2} = \frac{2y}{2} = y$$

so $u_{xx} = 4x^2 u_{\bar r\bar r} + 2u_{\bar r}$ $\quad u_y = u_{\bar s}$

$u_{yy} = u_{\bar s\bar s}$

so $4x^2 u_{\bar r\bar r} + 2u_{\bar r} - 4x^2 u_{\bar s\bar s} + 2u_{\bar s} = 0$

$$u_{\bar r\bar r} - u_{\bar s\bar s} + \frac{u_{\bar r} + u_{\bar s}}{2x^2} = 0$$

$$u_{\bar r\bar r} - u_{\bar s\bar s} + \frac{u_{\bar r} + u_{\bar s}}{2\bar r} = 0$$

Is there a better way?

we need to find r & s by solving

$ar_x^2 + br_xr_y + cr_y^2 = 0 \qquad aS_x^2 + bS_xS_y + cS_y^2 = 0$

$a\left(\frac{r_x}{r_y}\right)^2 + b\frac{r_x}{r_y} + c = 0 \qquad a\left(\frac{S_x}{S_y}\right)^2 + b\frac{S_x}{S_y} + c = 0$

$\frac{r_x}{r_y} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad \frac{S_x}{S_y} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

inherit here is the +/-.

Previous example

$u_{xx} - 4x^2 u_{yy} + 2u_y = 0$

$r_x^2 - 4x^2 r_y^2 = 0$

$\left(\frac{r_x}{r_y}\right)^2 - 4x^2 = 0 \qquad \frac{r_x}{r_y} = \pm 2x$

$r_x - (\pm 2x) r_y = 0 \qquad \frac{dx}{1} = \frac{dy}{-(\pm 2x)}\ ; \ dr = 0$

$2x\,dx = \pm dy\ ; \ dr = 0 \qquad r = R(x^2 \pm y)$

$c_1 = x^2 \pm y \quad r = c_2 \qquad r = x^2 \quad s = y$ we saw this already