Math 4381/6378 Sym

last class we considered

$$y'' + yy' + y^3 = 0$$

and showed that it has 2 symmetries

$$X = c_1 x + c_2, \quad Y = -c_1 y$$

So what do we do with these.

with 1st order we introduced new variable (r, s) such that

$$X r_x + Y r_y = 0 \qquad X s_x + Y s_y = 1$$

and in terms of the new variable we obtained a separable ODE

$$\frac{ds}{dr} = G(r)$$

Let's see what happens if we do the same here.

$\underline{\Gamma_1 = \frac{\partial}{\partial x}}$ $\quad (X = 1, Y = 0)$

so we solve

$$r_x = 0 \qquad s_x = 1$$

$$r = R(y) \qquad s = x + S(y)$$

choose $r = y$, $s = x$ or $x = s$, $y = r$

$$\frac{dy}{dx} = \frac{\frac{dy}{dr}}{\frac{dx}{dr}} = \frac{1}{s'}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dr}\left(\frac{1}{s'}\right)}{\frac{dx}{dr}} = \frac{-\frac{s''}{s'^2}}{s'} = -\frac{s''}{s'^3}$$

sub $y'' + yy' + y^3 = 0$

$$-\frac{s''}{s'^3} + \frac{r}{s'} + r^3 = 0 \quad \text{or} \quad s'' = rs'^2 + r^3 s'^3$$

no s

if we let $u = s'$ then $u' = s''$ and the ODE becomes

$$u' = ru^2 + r^3u^3 \quad 1^{st} \text{ order}$$

$$\underline{\Gamma_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}} \qquad (X = x,\ Y = y)$$

so we solve

$$x r_x - y r_y = 0 \qquad x s_x - y s_y = 1$$

$$r = R(xy) \qquad s = \ln x + \tilde{S}(xy)$$

choose $r = xy$ $\quad s = \ln x$ or $x = e^{s}$ $\quad y = r e^{-s}$

$$y' = \frac{\frac{d}{dr}(re^{-s})}{\frac{d}{dr}e^{s}} = \frac{e^{-s} - re^{-s}s'}{e^{s}s'} = e^{-2s}\left(\frac{1}{s'} - r\right)$$

$$y'' = e^{-3s}\left(\frac{-s'' - 3rs'^2 + 2rs'^3}{s'^3}\right)$$

& $y'' + yy' + y^3 = 0$ becomes

$$s'' + (r^2 - r^3 - 2r)s'^3 + (3 - r)s'^2 = 0$$

if $u = s'$ $\quad u' = s''$

& we get $\quad u' + (r^2 - r^3 - 2r)u^3 + (3 - r)u^2 = 0 \quad$ 1st order ODE.

so we ask. - is it possible to somehow use the symmetry we didn't use. In each case let's see what happens to the other symmetry

Case I we used $X=1,\ Y=0$

what happens to $X=x,\ Y=y$

the change of variables is

$$r=y,\quad s=x,\quad \&\quad u=s'$$

Sym Generator old System

$$\Gamma = X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+Y[x]\frac{\partial}{\partial y'},$$

& new system

$$\Gamma^{*}= R\frac{\partial}{\partial r}+S\frac{\partial}{\partial s}+S[r]\frac{\partial}{\partial s'} \qquad 2^{nd} \text{ order ODE}$$

& $$\bar{\Gamma} = R\frac{\partial}{\partial r}+U\frac{\partial}{\partial u} \qquad 1^{st} \text{ order ODE}$$

Note: $Y[x]= Y_x+(Y_y-X_x)y'-X_y y'^2,\quad S[r]= S_r+(S_s-R_r)s'-R_s s'^2$

if $X = x,\ Y = -y$

then if $r = y$ then $R = Y = -y = -r$

and $s = x$ so $S = X = x = s$

$$\bar{U} = S_r + (S_s - R_r)s' - R_s s'^2$$

$$= 0 + (1+1)s' - 0$$

$$= 2s' = 2u \quad (\text{note } u = s')$$

so 2nd symmetry becomes

$$R = -r, \quad \bar{U} = 2u \qquad (A)$$

Lie Group $\bar{r} = e^{-\varepsilon} r, \quad \bar{u} = e^{2\varepsilon} u$

$$\frac{d\bar{u}}{d\bar{r}} = e^{3\varepsilon} \frac{du}{dr} \quad \& \quad \bar{u}' = \bar{r}\bar{u}^2 + \bar{r}^3 \bar{u}^3$$

$$\cancel{e^{3\varepsilon}} \frac{du}{dr} = \cancel{e^{3\varepsilon}} r u^2 + \cancel{e^{-3\varepsilon}} r^3 \cancel{e^{6\varepsilon}} u^3$$

so invariant & we can use (A) sym

to reduce 1st order ODE to one that is separable

Case 2 we used $X = x$, $Y = -y$

what happens to $X = 1$, $Y = 0$

the change of variables is

$$r = xy \qquad s = \ln x \qquad (x = e^{s},\ y = r e^{-s})$$

so $R = Xy + xY$ $\qquad S = \frac{1}{x} X$

$= y \qquad\qquad = \frac{1}{x}$

$R = r e^{-s} \qquad S = e^{-s}$

$$\text{so } \mho = S_{[r]} = S_r + (S_s - R_r)s' - R_s s'^2$$

$$= 0 + (-e^{-s} - e^{-s})s' + r e^{-s} s'^2$$

$$= -2e^{-s}s' + r e^{-s} s'^2$$

Now $u = s'$

$$\mho = -2e^{-s}u + r e^{-s}u^2 \qquad \text{what } s?$$

$u = s'$ so $s = \int u\,dr$ $\qquad \mho = (-2u + ru^2)\, e^{-\int u\,dr}$

non local variable

so we see that if we use

$X = 1, \; Y = 0$

then $X = x, \; Y = -y$ gets passed to 1st order ODE

if we use

$X = x, \; Y = -y$

then $X = 1, \; Y = 0$ does not get passed.

So how do we know which one to use.?

<u>Lie Bracket</u>

~~If~~ If $\Gamma_1 = X_1 \frac{\partial}{\partial x} + Y_1 \frac{\partial}{\partial y}$ $\quad \Gamma_2 = X_2 \frac{\partial}{\partial x} + Y_2 \frac{\partial}{\partial y}$

we define the Lie Bracket as

$$[\Gamma_1, \Gamma_2] = \Gamma_1(\Gamma_2) - \Gamma_2(\Gamma_1)$$

if $[\Gamma_1, \Gamma_2] = k\Gamma_1$ if we use Γ_1 Γ_2 gets passed.

k const

(could be 0)

ex $\Gamma_1 = \frac{\partial}{\partial x}$ $\Gamma_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$

$$\Gamma_1(\Gamma_2 f) = \frac{\partial}{\partial x}(x f_x - y f_y) = f_x + x f_{xx} - y f_{xy}$$

$$\Gamma_2(\Gamma_1 f) = \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) f_x = x f_{xx} - y f_{xy}$$

$$\Gamma_1(\Gamma_2 f) - \Gamma_2(\Gamma_1 f) = f_x + x f_{xx} - y f_{xy} - (x f_{xx} - y f_{xy})$$

$$= f_x = \Gamma_1 f$$

so $[\Gamma_1, \Gamma_2] = \Gamma_1$

so use Γ_1 & Γ_2 will pass (we saw this)

Ex 2 The ODE

$$y'' + \frac{y'}{x} = e^{y}$$

has 2 sym. with

$$X = c_1 x + c_2 x \ln x, \quad Y = -2c_1 - 2c_2 - 2c_2 \ln x$$

Define $\Gamma_1 = x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}$ $\quad \Gamma_2 = x\ln x \frac{\partial}{\partial x} - 2(1+\ln x)\frac{\partial}{\partial y}$

$[\Gamma_1, \Gamma_2] = \Gamma_1$ so use Γ_1 & poss Γ_2

so $\quad x r_x - 2 r_y = 0 \qquad x s_x - 2 s_y = 1$

$$\frac{dx}{x} = \frac{dy}{-2}\ ; \ dr = 0 \qquad \frac{dx}{x} = \frac{dy}{-2} = \frac{ds}{1}$$

$r = R(2\ln x + y) \qquad s = \ln x + S(2\ln x + y)$

$r = 2\ln x + y \qquad s = \ln x$

New ODE

$y = r - 2s \qquad x = e^{s} \qquad s'' + e^{r} s'^{3} = 0$

$s' = u$ so $\quad u' + e^{r} u^{3} = 0 \quad$ already separable

but let's continue

let's see what happens to

$$X = x\ln x \qquad Y = -2 - 2\ln x$$

C of V $\quad r = 2\ln x + y \qquad s = \ln x$

$$R = \frac{2}{x} X + Y, \qquad S = \frac{1}{x} X$$

$$= \frac{2}{\cancel{x}} \cdot \cancel{x}\ln x - 2 - 2\ln x \qquad = \frac{x\ln x}{x}$$

$$= \ln x$$

$$= -2$$

so $\quad R = -2 \quad S = s \quad \bar{U} = S[r] = S_r + (S_s - R_r)s' - R_s s'^2$

$$= s' = u$$

so new sym

$$R = -2, \ \bar{U} = u \qquad \bar{u} = e^{\varepsilon} u, \ \bar{r} = r - 2\varepsilon$$

$$\frac{d\bar{u}}{d\bar{r}} = e^{\varepsilon}\frac{du}{dr} \qquad \bar{u}' + e^{\bar{r}}\bar{u}^{3} = 0 \qquad e^{\varepsilon}\frac{du}{dr} + e^{r-2\varepsilon}\cdot e^{3\varepsilon} u = 0$$

so invariant

New variables p, q

$$-2 p_r + u p_u = 0 \qquad -2 q_r + u q_u = 1$$

$$\frac{dr}{-2} = \frac{du}{u} ; dp = 0 \qquad \frac{dr}{-2} = \frac{du}{u} = \frac{dq}{1}$$

$$-\frac{1}{2}\cancel{r} = \ln u + c_1$$

$$c_1 = \ln u - r/2$$

$$p = P(2\ln u + r/\cancel{4}) \qquad q = -\frac{r}{2} + Q(2\ln u - r/\cancel{4})$$

$$p = 2\ln u + r \qquad q = -r/2$$

$$r = -2q$$

$$u = e^{\frac{p+2q}{2}}$$

$$u_r = \frac{e^{\frac{p+2q}{2}}\left(\frac{1}{2}+q'\right)}{-2q'} \qquad \frac{e^{\frac{p+2q}{2}}\left(\frac{1}{2}+q'\right)}{-2q'} + e^{-2q}\, e^{3\left(\frac{p+2q}{2}\right)}$$

$$\frac{\frac{1}{2}+q'}{-2q'} + e^{-2\cancel{q} + \frac{3p}{2} + 3\cancel{q} - p/2 - \cancel{q}} = 0 \qquad \frac{\frac{1}{2}+q'}{-2q'} + e^{p} = 0$$