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START-UP FLOWS OF SECOND GRADE FLUIDS IN DOMAINS WITH ONE FINITE DIMENSION

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Abstract—A number of unidirectional transient flows of a second grade fluid in a domain with one finite dimension are studied. The method of integral transforms (Fourier, Hankel or Laplace) is applied to obtain exact solutions. A general theorem on start-up flows for second grade fluids is presented that allows us to determine unidirectional flows of second grade fluids once the corresponding solution is known within the context of the Navier–Stokes theory. In the process of obtaining solutions for the fluid of second grade, we find several new exact solutions within the context of the classical Navier–Stokes theory.

1. INTRODUCTION

Recently, there have been several rigorous mathematical papers devoted to the study of existence, uniqueness, and stability of solutions for an incompressible homogeneous second grade fluid (cf. Galdi *et al.* [1], Galdi and Sequeira [2], Coscia and Wideman [3], Galdi *et al.* [4], Dunn and Fosdick [5] and Fosdick and Rajagopal [6]). Dunn and Fosdick [5] carried out an extensive study of the thermomechanics of fluids of second grade. They showed that if all the motions of the fluid meet the Clausius–Duhem inequality and the assumption that the specific Helmholtz free energy be a minimum in equilibrium, then the material coefficients obey $\mu \geq 0$, $\alpha_1 > 0$ and $\alpha_1 + \alpha_2 = 0$.

As early as 1963, Ting [7] provided a set of exact solutions for start-up flows of second grade fluids and recognized that the material constants α_1 and α_2 are subject to specific restrictions if the solutions are to be bounded. In fact, Ting [7] showed that if the condition that $\alpha_1 > 0$ is violated, then the solutions become unbounded. Coleman *et al.* [8] also studied an unsteady unidirectional flow and showed that the solutions blow up in time if $\alpha_1 > 0$. A recent paper by Dunn and Rajagopal [9] discusses in detail various issues concerning the thermomechanics of fluids of the differential type.

The purpose of this work is to study the general mixed initial-boundary value problem governing unidirectional unsteady flows involving second grade fluids with a view to emphasize the differences between the unsteady flow of a second grade fluid and the corresponding flow of a classical viscous fluid. We have found several new exact solutions to flows that might be relevant to problems in physics and engineering, and useful in the experimental determination of the material constants α_1 and α_2. In this paper we study the following problems:

(a) formation of Couette flow between parallel plates;
(b) flow between two parallel plates due to an impulsive body force or pressure gradient;
(c) Taylor–Couette and Couette flows in an annulus due to a constant velocity suddenly applied to one of the boundaries;
(d) Taylor–Couette and Couette flows in an annulus due to a constant shear suddenly applied to one of the boundaries.

Many more exact solutions can be determined; instead of listing them one after the other, we will present a theorem on unidirectional transient flows for second grade fluids which

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makes a systematic study redundant; in fact, we will show that the velocity profile is immediately derivable from the corresponding one for the Navier–Stokes fluid. We also consider unidirectional flows due to the application of impulsive pressure gradients or body forces for which our theorem does not apply.

2. GOVERNING EQUATIONS

The incompressible second grade fluid is characterized by the following constitutive equation (cf. [10]):

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \tag{2.1}$$

where μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli; $-pI$ denotes the indeterminate spherical stress and A_1 and A_2 are the kinematic tensors defined through

$$A_1 = (\operatorname{grad} v) + (\operatorname{grad} v)^T, \tag{2.2a}$$

and

$$A_2 = \frac{\mathrm{d}}{\mathrm{d}t}A_1 + A_1(\operatorname{grad} v) + (\operatorname{grad} v)^T A_1. \tag{2.2b}$$

Here v is the velocity, grad the gradient operator and $(\mathrm{d}/\mathrm{d}t)$ the material time derivative.

If we substitute the stress T into the balance of linear momentum

$$\operatorname{div} T + \rho b = \rho\frac{\mathrm{d}v}{\mathrm{d}t}, \tag{2.3}$$

we obtain, in the case of a conservative body force field $b = -\operatorname{grad}\phi$,

$$\mu\Delta v + \alpha_1\Delta v_t + \alpha_1(\Delta w \times v) + (\alpha_1 + \alpha_2)\{A_1\Delta v + 2\operatorname{div}[(\operatorname{grad} v)(\operatorname{grad} v)^T]\} - \rho(w \times v) - \rho v_t = \operatorname{grad} P, \tag{2.4}$$

where

$$P = p - \alpha_1(v\cdot\Delta v) - \frac{(2\alpha_1 + \alpha_2)}{4}|A_1|^2 + \frac{1}{2}\rho|v|^2 + \rho\phi, \tag{2.5}$$

and Δ is the Laplacian, the subscript t indicates partial differentiation with respect to time, $|A_1|$ the trace norm of A_1 and

$$w = \operatorname{curl} v. \tag{2.6}$$

As the fluid in incompressible, it can undergo only isochoric motions; therefore

$$\operatorname{div} v = 0. \tag{2.7}$$

If the domain R is bordered by two non-intersecting boundaries S_1, S_2 and we assume a unidirectional motion of the form

$$v = v(x, t)j, \tag{2.8}$$

in Cartesian coordinates,

$$v = v(x, t)e_z, \tag{2.9}$$

or, alternatively,

$$v = v(x, t)e_\theta, \tag{2.10}$$

in cylindrical coordinates, and

$$v = v(x, t)\sin\theta e_\theta, \tag{2.11}$$

in spherical coordinates (x is the spatial coordinate on which the velocity depends, e_θ and e_z are, respectively, the unit vectors in the θ- and the z-directions), then it can be shown that the general form of the balance of linear momentum is

$$\mu\left[\frac{1}{x^p}\frac{\partial}{\partial x}\left(x^p\frac{\partial v}{\partial x}\right)-n\frac{v}{x^2}\right]+\alpha_1\frac{\partial}{\partial t}\left[\frac{1}{x^p}\frac{\partial v}{\partial x}\left(x^p\frac{\partial v}{\partial x}\right)-n\frac{v}{x^2}\right]+g(x,t)=\rho\frac{\partial v}{\partial t},\quad x\in R,\quad t>0. \tag{2.12}$$

In (2.12), $n = 1$ when the velocity has the form in (2.8), otherwise $n = 0$; $p = 0, 1, 2$, for Cartesian, cylindrical, and spherical coordinates, respectively (no summation on p is intended). Here $g(x, t)$ represents the sum of the body forces and the pressure gradient. The linear partial differential equation (2.12) is to be solved under the boundary conditions

$$q_i\left[\mu\left(\frac{\partial v}{\partial x}-\gamma\frac{v}{x}\right)+\alpha_1\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\gamma\frac{v}{x}\right)\right]+(1-q_i)v=f_i(t)\quad x\in S_i,\quad t>0, \tag{2.13}$$

where $\gamma = 1$ when the velocity assumes the forms in (2.10) and (2.11), otherwise $\gamma = 0$. ($i = 1, 2$; also if $q_i = 0$ we call it a boundary condition at S_i of the first kind and if $q_i = 1$ we call it a boundary condition of the second kind.) The initial condition is

$$v=F(x),\quad x\in R,\quad t=0, \tag{2.14}$$

We will give the solution to the mixed initial-boundary value problem represented by (2.12)–(2.14) by employing a finite integral transform to eliminate the spatial variable and the Laplace transform to eliminate the time variable. Then we will provide the explicit solution for flows of problems of engineering and physical interest by using the Laplace transform. Problems for which the boundary data are incompatible do not admit smooth solutions that satisfy both the initial and boundary conditions (cf. Bandelli *et al.* [22]). The ones here examined do yield smooth solutions.

3. THE GENERAL GOVERNING EQUATION FOR UNIDIRECTIONAL MOTION OF A SECOND GRADE FLUID

In general, it is not convenient to use the method of separation of variables to solve non-homogeneous linear partial differential equations. In this case the integral transform technique presents a systematic, efficient and powerful tool.

3.1. *Development of the integral–transform pair*

An integral–transform pair can be developed by considering the eigenvalue problem

$$\nabla^2\psi-\left(\frac{n}{x^2}-\lambda^2\right)\psi=0\quad x\in R, \tag{3.1}$$

$$q_i\left(\frac{\partial\psi}{\partial x}-\gamma\frac{\psi}{x}\right)+(1-q_i)\psi=0\quad x\in S_i, \tag{3.2}$$

where $\psi = \psi(\lambda, x)$. Notice that the following identity is valid

$$\frac{1}{x^p}\frac{\partial}{\partial x}\left(x^p\frac{\partial v}{\partial x}\right)=\nabla^2 v,$$

$p = 0, 1, 2$, for Cartesian, cylindrical, and spherical coordinates, respectively (no summation on p is intended). We define the integral–transform pair in the spatial variable x for a function $f(x, t)$ as:

$$\text{integral transform: } \bar{f}(\lambda_m, t)=\int_R \psi(\lambda_m, x)f(x,t)\,\mathrm{d}v, \tag{3.3}$$

$$\text{inversion formula: } f(x,t)=\sum_m\frac{\psi(\lambda_m, x)}{N(\lambda_m)}\bar{f}(\lambda_m, t), \tag{3.4}$$

where $\bar{f}(\lambda_m, t)$ is called integral transform of the function $f(x, t)$ with respect to the variable x, and

$$N(\lambda_m) = \int_R [\psi(\lambda_m, x)]^2 \, \mathrm{d}v.$$

3.2. *Integral transform of the balance of linear momentum*

Let us multiply both sides of (2.12) by $\psi(\lambda_m, x)$ and integrate over the region R. Then,

$$\int_R \mu\psi_m(x)\nabla^2 v(x, t)\,\mathrm{d}v - n\int_R \mu \frac{v(x, t)\psi_m(x)}{x^2}\mathrm{d}v + \alpha_1 \frac{\partial}{\partial t}\int_R \psi_m(x)\nabla^2 v(x, t)\,\mathrm{d}v$$
$$- n\alpha_1 \frac{\partial}{\partial t}\int_R \frac{v(x, t)\psi_m(x)}{x^2}\,\mathrm{d}v + \int_R \psi_m(x) g^*(x, t)\,\mathrm{d}v = \rho\frac{\partial}{\partial t}\int_R \psi_m(x) v(x, t)\,\mathrm{d}v, \tag{3.5}$$

where $\psi_m(x)$ indicates $\psi(\lambda_m, x)$. Using Green's theorem

$$\int_R \psi_m(x)\nabla^2 v(x, t)\,\mathrm{d}v = \int_R v(x, t)\nabla^2\psi_m(x)\,\mathrm{d}v + \sum_{i=1}^{2}\int_{S_i}\left(\psi_m(x)\frac{\partial v(x, t)}{\partial x} - v(x, t)\frac{\mathrm{d}\psi_m(x)}{\mathrm{d}x}\right)\mathrm{d}s, \tag{3.6}$$

and (3.5), we have

$$\mu\int_R v\nabla^2\psi_m\,\mathrm{d}v - n\mu\int_R \frac{v\psi_m}{x^2}\mathrm{d}v + \alpha_1\frac{\partial}{\partial t}\int_R v\nabla^2\psi_m\,\mathrm{d}v - n\alpha_1\frac{\partial}{\partial t}\int_R \frac{v\psi_m}{x^2}\mathrm{d}v$$
$$+ \sum_{i=1}^{2}\int_{S_i}\psi_m\left(\mu\frac{\partial v}{\partial x} + \alpha_1\frac{\partial}{\partial t}\frac{\partial v}{\partial x}\right)\mathrm{d}s - \sum_{i=1}^{2}\int_{S_i}\frac{\mathrm{d}\psi_m}{\mathrm{d}x}\left(\mu v + \alpha_1\frac{\partial v}{\partial t}\right)\mathrm{d}s + \int_R \psi_m g^m\mathrm{d}v = \rho\frac{\partial}{\partial t}\int_R \psi_m v\,\mathrm{d}v. \tag{3.7}$$

Let us multiply each side of (3.1) by $v(x, t)$ and integrate over the region R to obtain

$$\int_R v\nabla^2\psi_m\,\mathrm{d}v - n\int_R \frac{v\psi_m}{x^2}\mathrm{d}v = -\lambda_m^2\int_R \psi_m v\,\mathrm{d}v = -\lambda_m^2\bar{v}(\lambda_m, t). \tag{3.8}$$

Then, substituting the result into (3.7), we get

$$-\lambda_m^2\left(\mu\bar{v}(\lambda_m, t) + \alpha_1\frac{\mathrm{d}\bar{v}(\lambda_m, t)}{\mathrm{d}t}\right) + \bar{g}(\lambda_m, t) + C_1(\lambda_m, t) + C_2(\lambda_m, t) = \rho\frac{\mathrm{d}\bar{v}(\lambda_m, t)}{\mathrm{d}t}, \tag{3.9}$$

where the functions $C_i(\lambda_m, t)$ are given by

$$q_i = 0 \to C_i(\lambda_m, t) = -\int_{S_i}\frac{\mathrm{d}\psi_m}{\mathrm{d}x}\left(\mu f_i + \alpha_1\frac{\partial f_i}{\partial t}\right)\mathrm{d}s,$$
$$q_i = 1, \qquad \gamma = 0 \to C_i(\lambda_m, t) = \int_{S_i}\psi_m f_i \mathrm{d}s, \qquad \gamma = 1 \to C_i(\lambda_m, t) = \int_{S_i} x\frac{\mathrm{d}\psi_m}{\mathrm{d}x} f_i \mathrm{d}s.$$

If we set

$$A(\lambda_m, t) = \bar{g}(\lambda_m, t) + C_1(\lambda_m, t) + C_2(\lambda_m, t),$$

and

$$A^*(\lambda_m, t) = \frac{A(\lambda_m, t)}{\rho + \alpha_1\lambda_m^2},$$

then (3.9) yields

$$\frac{\mathrm{d}\bar{v}(\lambda_m, t)}{\mathrm{d}t} + \frac{\mu\lambda_m^2}{\rho + \alpha_1\lambda_m^2}\bar{v}(\lambda_m, t) = A^*(\lambda_m, t). \tag{3.10}$$

3.3. *Solution of the ordinary differential equation and inversion*

We can solve (3.10) for $\bar{v}(\lambda_m, t)$ as

$$\bar{v}(\lambda_m, t) = \int A^*(\lambda_m, t)\exp\left\{\frac{\mu\lambda_m^2}{\rho + \alpha_1\lambda_m^2}(\tau - t)\right\}\mathrm{d}\tau + C\exp\left\{-\frac{\mu\lambda_m^2}{\rho + \alpha_1\lambda_m^2}t\right\}. \quad (3.11)$$

The constant C can be found by using the initial condition $\bar{v}(\lambda_m, 0) = F(\lambda_m)$, and hence

$$\bar{v}(\lambda_m, t) = \int_0^t \frac{A(\lambda_m, \tau)}{\rho + \alpha_1\lambda_m^2}\exp\left\{\frac{\mu\lambda_m^2}{\rho + \alpha_1\lambda_m^2}(\tau - t)\right\}\mathrm{d}\tau + F(\lambda_m)\exp\left\{-\frac{\mu\lambda_m^2}{\rho + \alpha_1\lambda_m^2}t\right\}. \quad (3.12)$$

Integrating by parts, we obtain

$$\bar{v}(\lambda_m, t) = \frac{1}{\mu\lambda_m^2}\left[A(\lambda_m, t) - A(\lambda_m, 0)\exp\left\{\frac{\mu\lambda_m^2}{\rho + \alpha_1\lambda_m^2}t\right\} - \int_0^t \frac{\mathrm{d}A}{\mathrm{d}\tau}\exp\left\{\frac{\mu\lambda_m^2}{\rho + \alpha_1\lambda_m^2}(\tau - t)\right\}\mathrm{d}\tau\right]$$
$$+ F(\lambda_m)\exp\left\{-\frac{\mu\lambda_m^2}{\rho + \alpha_1\lambda_m^2}t\right\}. \quad (3.13)$$

Now suppose that $A(x, t)$ is independent of the spatial variable and has the form

$$A(x, t) = \sum_{i=1}^{\infty} A_i H_i(t - t_i),$$

where A_i are constant and $H_i(t - t_i)$ is the Heaviside step function with step at $t - t_i$. In doing so we implicitly allow for the body forces, the pressure gradient, the boundary conditions or any combination of them to undergo only step changes in time. Let us restrict ourselves to the case when

$$A(x, t) = AH(t),$$

then

$$\bar{v}(\lambda_m, t) = \frac{A}{\mu\lambda_m^2}\left[1 - \exp\left\{-\frac{\mu\lambda_m^2}{\rho + \alpha_1\lambda_m^2}t\right\}\right] + F(\lambda_m)\exp\left\{-\frac{\mu\lambda_m^2}{\rho + \alpha_1\lambda_m^2}t\right\}, \quad t > 0. \quad (3.14)$$

With the help of the inversion formula (3.3) we obtain the final expression for the velocity

$$v(x, t) = \frac{A}{\mu}\sum_m \frac{\psi(\lambda_m, x)}{\lambda_m^2 N(\lambda_m)} + \sum_m \frac{\psi(\lambda_m, x)}{N(\lambda_m)}\left(\frac{A}{\mu\lambda_m^2} + F(\lambda_m)\right)\exp\left\{-\frac{\mu\lambda_m^2}{\rho + \alpha_1\lambda_m^2}t\right\}, \quad t > 0. \quad (3.15)$$

The first term on the right-hand side of (3.15) represents the steady state velocity, the second term the transient response of the flow to an abrupt change either in the boundary conditions or body forces or pressure gradient. It is worthwhile pointing out that the boundary conditions implicitly determine the kind of flow taking place because they affect the form of the functions $C_i(\lambda_m, t)$ and therefore of A. We are now in the position to state the following:

Theorem. If a second grade fluid is at rest or is undergoing steady unidirectional motion in a bounded domain and, at some instant, the boundary conditions, the body forces, the pressure gradient, or any combination thereof experience a step change, then the resulting transient motion will be the same as that of a purely viscous fluid but with characteristic time $t_m = (\rho + \alpha_1\lambda_m^2)/\mu\lambda_m^2$, where λ_m are the eigenvalues of the Sturm–Liouville problem associated with the mixed initial-boundary value problem governing the flow of the Navier–Stokes fluid, provided that the body forces are independent of the position.

To illustrate the power of this theorem we shall consider two similar problems: the axial Couette and the Taylor–Couette flow formation in an annulus due to the imposition of a constant shear. The ease with which the latter can be solved, once the Navier–Stokes solution is known, will become self-evident.

4. AXIAL COUETTE FLOW FORMATION IN AN ANNULUS DUE TO A LONGITUDINAL CONSTANT SHEAR

Consider a second grade fluid at rest in an annular region. At time $t = 0^+$ *let the inner cylinder of radius* R_0 *be pulled with constant shear along its axis and the outer one of radius* R_1 *be held fixed.*

Let us assume a velocity field of the form

$$\mathbf{w} = (0, 0, w(r, t)).$$

The balance of linear momentum yields

$$\mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \alpha_1 \frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \rho\frac{\partial w}{\partial t} = 0, \quad R_0 < r < R_1, \quad t > 0, \tag{4.1}$$

and the boundary conditions are

$$w = 0, \quad r = R_1, \quad t > 0, \tag{4.2}$$

$$\tau_{rz} = \mu\frac{\partial w}{\partial r} + \alpha_1 \frac{\partial^2 w}{\partial t \partial r} = f, \quad r = R_0, \quad t > 0, \tag{4.3}$$

with initial condition

$$w = 0, \quad R_0 \le r \le R_1, \quad t = 0. \tag{4.4}$$

The subsidiary equation obtained by applying the Laplace transform to (4.1) and using the initial condition (4.4) is

$$\frac{\mathrm{d}^2 \bar{w}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\bar{w}}{\mathrm{d}r} - \frac{\rho p}{\mu + \alpha_1 p}\bar{w} = 0, \quad R_0 < r < R_1, \tag{4.5}$$

whose solution, under the transformed boundary conditions

$$\bar{w} = 0, \qquad r = R_1, \tag{4.6}$$

$$\frac{\mathrm{d}\bar{w}}{\mathrm{d}r} = \frac{f}{p(\mu + \alpha_1 p)}, \qquad r = R_0, \tag{4.7}$$

is

$$\begin{aligned}\bar{w} &= \frac{f}{p\sqrt{(\mu + \alpha_1 p)\rho p}} \frac{I_0(zr)K_0(zR_1) - I_0(zR_1)K_0(zr)}{I_1(zR_0)K_0(zR_1) + I_0(zR_1)K_1(zR_0)} \\ &= \frac{f}{\rho}\frac{z}{p^2}\frac{I_0(zr)K_0(zR_1) - I_0(zR_1)K_0(zr)}{I_1(zR_0)K_0(zR_1) + I_0(zR_1)K_1(zR_0)}\end{aligned} \tag{4.8a}$$

(the bar denotes the Laplace transform of w) where

$$z = \left(\frac{\rho p}{\mu + \alpha_1 p}\right)^{1/2}. \tag{4.8b}$$

We shall choose the branch of z such that $-\pi/2 < \arg z \le \pi/2$. Clearly, the singularities of (4.8a) are the zeros of the denominator. It is easy to show that $p = 0$ is a simple pole; to find the other poles, let us set $z = i\lambda$ into the denominator of (4.8a) and compute the roots of the transcendental equation

$$I_1(i\lambda R_0)K_0(i\lambda R_1) + I_0(i\lambda R_1)K_1(i\lambda R_0) = -\frac{1}{2}\pi i[J_1(\lambda R_0)Y_0(\lambda R_1) - J_0(\lambda R_0)Y_1(\lambda R_1)] = 0. \tag{4.9}$$

In (4.9) we have used the identities (cf. [11])*

$$I_0(ix) = J_0(x), \tag{4.10a}$$

$$I_1(ix) = iJ_i(x), \tag{4.10b}$$

$$K_0(ix) = -\tfrac{1}{2}\pi i[J_0(x) - iY_0(x)], \tag{4.10c}$$

$$K_1(ix) = -\tfrac{1}{2}\pi[J_1(x) - iY_1(x)], \tag{4.10d}$$

The theorem in the Appendix ensures that there are infinitely many roots of (4.9) and that they are simple, real and therefore, they can be ordered to form an increasing sequence $\{\lambda_n\}$. Thus the singularities of $\bar{w}$ are

$$p_0 = 0, \qquad p_n = -\frac{\mu\lambda_n^2}{\rho + \alpha_1\lambda_n^2}, \quad (n = 1, 2, \ldots), \tag{4.11}$$

[λ_n are the positive roots of (4.9)] and are concentrated in the interval $-\mu/\alpha_1 \leq p \leq 0$, with $p = -\mu/\alpha_1$ as the point of accumulation. Hence $\bar{w}$ is a regular analytic function of p if $Re\, p > 0$. Direct computation gives the residues R_n at the poles p_n:

$$R_0 = \frac{f}{\mu} R_0 \log\frac{r}{R_1}, \tag{4.12}$$

$$R_n = -\frac{\pi f}{\mu} \frac{J_0(\lambda_n r) Y_0(\lambda_n R_1) - J_0(\lambda_n R_1) Y_0(\lambda_n r)}{\lambda_n [J_0^2(\lambda_n R_1) - J_1^2(\lambda_n R_0)]} J_0(\lambda_n R_1) J_1(\lambda_n R_0) e^{p_n t}, \quad (n = 1, 2, \ldots). \tag{4.13}$$

The inverse transform of (4.8a) will be computed with the complex inversion formula (cf. [12])

$$w(r, t) = \frac{1}{2\pi i} \int_L \bar{w} e^{pt}\, dp, \tag{4.14}$$

where L is a path defined by $Re\, p =$ const. > 0 such that all the singularities of $\bar{w}$ are to the left of L.

At this juncture it would be appropriate to document results due to Ting [7] that will be useful in our analysis. The complex inversion integral (4.14) can be evaluated by applying the Cauchy residue theorem. However, an extension of this theorem is necessary because the integrand in (4.14) possesses infinitely many singularities concentrated in the interval $[\mu/\alpha_1, 0]$ and $p = -\mu/\alpha_1$ is an essential singularity. Towards this purpose, we consider the integral

$$\bar{w}^* = \frac{1}{2\pi i} \int_{L + C + C_N} \bar{w} e^{pt}\, dp. \tag{4.15}$$

The integration contour is reported in Fig. 1. Γ is a half-circle of radius R to the left of the path L and C_N a curve enclosing all the singularities p_n for $n > N$; Γ and C_N are connected by two portions of straight lines parallel to the real axis where $Re\, p < -\mu/\alpha_1$. The curves C_N are constructed so that they do not cut any singularity.

By the residue theorem, for all finite N

$$\frac{1}{2\pi i} \int_{L + C + C_N} \bar{w} e^{pt}\, dp = \sum_{n=0}^{N} R_n, \tag{4.16}$$

where R_n is the residue of $\bar{w}(p, t)$ at the pole p_n. We will show later that the line integrals over Γ and C_N vanish, respectively, as R and N tend to infinity, whereas those on the horizontal straight lines cancel each other, as the integrand is continuous there.

* *Op. cit.*, Carslaw and Jaeger, p. 351, formulae (30)–(33).

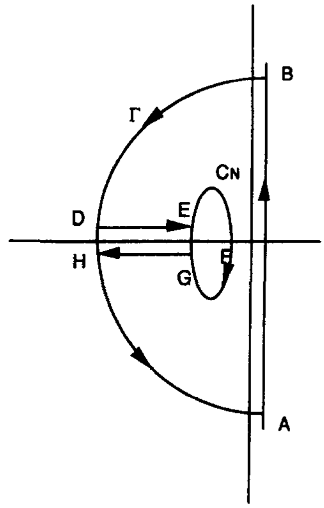

Fig. 1. Contour of integration.

By taking the limit of (4.16) as N and R go to infinity, we will formally have

$$w(r, t) = \frac{1}{2\pi i}\int_L \bar{w}e^{pt}\,dp = \sum_{n=0}^{\infty} R_n. \tag{4.17}$$

The formal solution obtained in (4.17) is to be checked *a posteriori* to ensure that it actually satisfies the given initial-boundary value problem.

To show that the complex inversion integral converges uniformly in r and t we need an estimate for $|\bar{w}|$ (cf. [12]).

The transformation $\rho p/(\mu + \alpha_1 p)$ maps the region $Re\,p > \alpha_0 > 0$ into a finite disk. Consequently, the conformal transformation

$$z = \left(\frac{\rho p}{\mu + \alpha_1 p}\right)^{1/2}, \quad -\frac{\pi}{2} < \arg z \leq \frac{\pi}{2},$$

maps the closed set $Re\,p > \alpha_0$ in the p-plane onto a closed half-disk F in the z-plane. Since

$$|z|\left|\frac{I_0(zr)K_0(zR_1) - I_0(zR_1)K_0(zr)}{I_1(zR_0)K_0(zR_1) + I_0(zR_1)K_1(zR_0)}\right|,$$

is a continuous function defined on the compact set $[R_0, R_1] \times F$, then by the Weirstrass theorem it must assume a maximum there. Thus, for $Re\,p \geq \alpha_0$,

$$\begin{aligned} |\bar{w}| &= \frac{f}{\rho}\frac{|z|}{|p|^2}\left|\frac{I_0(zr)K_0(zR_1) - I_0(zR_1)K_0(zr)}{I_1(zR_0)K_0(zR_1) + I_0(zR_1)K_1(zR_0)}\right| \\ &\leq \frac{\text{const.}}{|p|^2}\max_{\substack{z\in F\\ R_0 \leq r \leq R_1}}\left|\frac{I_0(zr)K_0(zR_1) - I_0(zR_1)K_0(zr)}{I_1(zR_0)K_0(zR_1) + I_0(zR_1)K_1(zR_0)}\right| \leq \frac{\text{const.}}{|p|^2}. \end{aligned} \tag{4.18}$$

Similar estimates for the successive derivatives of the integral (4.14) with respect to the space variable show that these are of the order of $O(1/|p|^2)$. Formulae (4.12), (4.13) and the residue theorem imply that

$$\begin{aligned} \frac{1}{2\pi i}\int_{L+C+C_N} \bar{w}(r, p)e^{pt}\,dp &= \frac{f}{\mu}R_0 \log\frac{r}{R_1} \\ &- \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)Y_0(\lambda_n R_1) - J_0(\lambda_n R_1)Y_0(\lambda_n r)}{\lambda_n[J_0^2(\lambda_n R_1) - J_1^2(\lambda_n R_0)]}J_0(\lambda_n R_1)J_1(\lambda_n R_0)e^{p_n t}. \end{aligned} \tag{4.19}$$

To prove that the integral (4.16) vanishes over Γ we observe that when $|p|$ is large enough

$$|p\bar{w}| = \frac{f}{\rho|p|}(\rho/\alpha_1)^{1/2}\left|\frac{I_0((\rho/\alpha_1)^{1/2}r)K_0((\rho/\alpha_1)^{1/2}R_1) - I_0((\rho/\alpha_1)^{1/2}R_1)K_0((\rho/\alpha_1)^{1/2}r)}{I_1((\rho/\alpha_1)^{1/2}R_0)K_0((\rho/\alpha_1)^{1/2}R_1) + I_0((\rho/\alpha_1)^{1/2}R_1)K_1((\rho/\alpha_1)^{1/2}R_0)}\right|$$

$$\leq \frac{\text{const.}}{|p|}. \tag{4.20}$$

that is, $p\bar{w}$ and, therefore, $\bar{w}$ uniformly tend to zero when the absolute value of p approaches infinity. Theorems on limiting contours in complex analysis (cf. [13])* ensure that (4.20) is a sufficient condition for the integral over Γ to vanish and for the initial condition to be met. If we let $N \to \infty$, (4.19) formally yields

$$w(r, t) = -\frac{1}{2\pi i}\lim_{N\to\infty}\int_{C_N}\bar{w}(r, p)\,e^{pt}\,dp + \frac{f}{\mu}R_0\log\frac{r}{R_1}$$
$$-\frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)Y_0(\lambda_n R_1) - J_0(\lambda_n R_1)Y_0(\lambda_n r)}{\lambda_n[J_0^2(\lambda_n R_1) - J_1^2(\lambda_n R_0)]}J_0(\lambda_n R_1)J_1(\lambda_n R_0)e^{p_n t}. \tag{4.21}$$

Let us verify that the series in (4.21) converges absolutely and uniformly in r and t and that it is a valid representation for $w(r, t)$. The theorem in the Appendix suggests that the roots λ_n of (4.9) are such that $\lambda_n/n = O(1)$ as $n \to \infty$. By virtue of the Bessel function expansions for large values of the argument

$$J_p(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4} - \frac{p\pi}{2}\right), \tag{4.22a}$$

$$Y_p(x) = \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4} - \frac{p\pi}{2}\right), \tag{4.22b}$$

we have

$$J_0(\lambda_n r)Y_0(\lambda_n R_1) - J_0(\lambda_n R_1)Y_0(\lambda_n r) \cong \frac{2}{\pi\lambda_n}\sqrt{\frac{1}{R_1 r}}\sin\lambda_n(R_1 - r) = O\left(\frac{1}{\lambda_n}\right), \tag{4.22c}$$

and

$$\frac{J_0(\lambda_n R_1)}{J_1(\lambda_n R_0)} \cong -\frac{1}{\sqrt{k}}\frac{\cos\left(\lambda_n R_1 - \frac{\pi}{4}\right)}{\sin\left(\lambda_n R_0 - \frac{\pi}{4}\right)} = O(1), \tag{4.22d}$$

as $n \to \infty$, where $k = R_1/R_0$ [observe that $J_0(\lambda_n R_1) \neq J_1(\lambda_n R_0)$ unless $R_1 = R_0$]. Therefore

$$\frac{J_0(\lambda_n R_1)J_1(\lambda_n R_0)}{J_0^2(\lambda_n R_1) - J_1^2(\lambda_n R_0)} = \frac{\dfrac{J_0(\lambda_n R_1)}{J_1(\lambda_n R_0)}}{\dfrac{J_0^2(\lambda_n R_1)}{J_1^2(\lambda_n R_0)} - 1} = O(1), \tag{4.23}$$

as $n \to \infty$; we conclude that $R_n = O(1/\lambda_n)^2$. Then, the convergence of the series is ensured.

To completely evaluate the inversion integral, following Ting [7] we will construct a series of closed curves C_N around point $p = -\mu/\alpha_1$, provided that they do not cut any pole, such that

$$\lim_{N\to\infty}\int_{C_N}\bar{w}(r, p)\,e^{pt}\,dp = 0. \tag{4.24}$$

After defining two sets of curves, respectively, in the z and p-plane by

$$z \equiv \tau\exp(i\eta), \qquad p + \frac{\mu}{\alpha_1} = \varepsilon\exp(i\theta), \tag{4.25}$$

*Op. cit., Hildebrand, pp. 589–590.

and setting

$$\tau \sin \eta = \lambda_N \pi \quad (N = 1, 2, \ldots), \tag{4.26}$$

it is easy to show that $\varepsilon_N = f(\theta, \lambda_N)$, where

$$\varepsilon_N = O\left(\frac{1}{\lambda_N}\right) \quad \text{as } N \to \infty. \tag{4.27}$$

If $\theta = 0$, we verify that no curve actually cuts any of the poles of $\bar{w}(r, p)$. Now

$$\left| \int_{C_N} \bar{w} e^{pt} dp \right| \leq \int_{-\pi}^{\pi} \frac{f}{\rho R_0} \frac{|z|}{|p|^2} \left| \frac{I_0(zr)K_0(zR_1) - I_0(zR_1)K_0(zr)}{I_1(zR_0)K_0(zR_1) + I_0(zR_1)K_1(zR_0)} \right| \varepsilon_N e^{|p|t} d\theta. \tag{4.28}$$

With the help of the asymptotic expansions for the modified Bessel functions valid for $|z| \gg 1$ (cf. [14])*

$$I_0(z) \cong \frac{1}{\sqrt{2\pi z}}(-ie^{-z} + e^z) \quad -\pi < \arg z \leq 0,$$

$$I_0(z) \cong \frac{1}{\sqrt{2\pi z}}(ie^{-z} + e^z) \quad 0 < \arg z \leq \pi,$$

$$I_1(z) \cong \frac{1}{\sqrt{2\pi z}}(ie^{-z} + e^z) \quad -\pi < \arg z \leq 0,$$

$$I_1(z) \cong \frac{1}{\sqrt{2\pi z}}(-ie^{-z} + e^z) \quad 0 < \arg z \leq \pi,$$

$$K_1(z) \cong \sqrt{\frac{\pi}{2z}} e^{-z} \quad -\pi < \arg z \leq \pi,$$

we find that, for the range $-\pi < \arg z \leq 0$,

$$N = I_0(zr)K_0(zR_1) - I_0(zR_1)K_0(zr) \cong \frac{1}{2z}\sqrt{\frac{1}{R_1 r}}\,(e^{z(r-R_1)} - e^{z(r-R_1)}), \tag{4.29}$$

$$D = I_1(zR_0)K_0(zR_1) + I_0(zR_1)K_1(zR_0) \cong \frac{1}{2z}\sqrt{\frac{1}{R_1 R_0}}\,(e^{z(R_0-R_1)} + e^{-z(R_0-R_1)}). \tag{4.30}$$

Hence

$$\frac{N}{D} \cong 2i\sqrt{\frac{R_0}{r}} \frac{\sinh z(r - R_1)}{\cosh z(R_0 - R_1)}, \tag{4.31}$$

and

$$\left|\frac{N}{D}\right| \leq 2\sqrt{\frac{R_0}{r}} \left| \frac{|e^{z(r-R_1)}| + |e^{-z(r-R_1)}|}{|e^{z(R_0-R_1)}| - |e^{-z(R_0-R_1)}|} \right|. \tag{4.32}$$

From (4.25) it follows immediately that

$$\tau \cos \theta = \lambda_N \pi \cot \theta$$

which, when used in (4.32), yields

$$\left|\frac{N}{D}\right| \leq \text{const} \frac{\cosh \lambda_N \pi \cot \theta (r - R_1)}{\sinh \lambda_N \pi \cot \theta (R_0 - R_1)} = O(1). \tag{4.33}$$

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*Op. cit., McLachlan, p. 201.

Ultimately

$$\left|\int_{C_N} \bar{w} e^{pt} \mathrm{d}p\right| \le \int_{-\pi}^{\pi} \mathrm{const} \frac{|z|}{|p|^2}\left|\frac{N}{D}\right| \varepsilon_N e^{pt} \mathrm{d}\theta$$

$$\le \int_{-\pi}^{\pi} \frac{\mathrm{const}}{|p|^{3/2}} \frac{1}{\sqrt{|\mu + \alpha_1 p|}} \varepsilon_N e^{|p|t} \mathrm{d}\theta \le \int_{-\pi}^{\pi} \frac{\mathrm{const}}{|p|^{3/2}} \varepsilon_N^{1/2} \mathrm{d}\theta = O\left(\frac{1}{\lambda_N}\right)^{1/2}.$$

Consequently, the limit (4.24) holds. For $0 \le \arg z < \pi$

$$N \cong \frac{1}{2z}\sqrt{\frac{1}{R_1 r}}(e^{z(r-R_1)} - e^{-z(r-R_1)}), \qquad D \cong \frac{1}{2z}\sqrt{\frac{1}{R_1 R_0}}(e^{z(R_0-R_1)} - e^{-z(R_0-R_1)}),$$

which are equal, respectively, to (4.29) and (4.30); hence, limit (4.24) holds throughout the whole complex plane. The final solution is

$$w(r,t) = \frac{f}{\mu} R_0 \log\frac{r}{R_1} - \frac{\pi f}{\mu}\sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)Y_0(\lambda_n R_1) - J_0(\lambda_n R_1)Y_0(\lambda_n r)}{\lambda_n[J_0^2(\lambda_n R_1) - J_1^2(\lambda_n R_0)]} J_0(\lambda_n R_1) J_1(\lambda_n R_0) e^{p_n t}, \tag{4.34}$$

or, on substitution of (4.9) into (4.34), we obtain

$$w(r,t) = \frac{f}{\mu} R_0 \log\frac{r}{R_1} + \frac{\pi f}{\mu}\sum_{n=1}^{\infty} \frac{J_0(\lambda_n R_1)[J_0(\lambda_n r)Y_1(\lambda_n R_1) - Y_0(\lambda_n r)J_1(\lambda_n R_1)]}{\lambda_n[J_1^2(\lambda_n R_0) - J_0^2(\lambda_n R_1)]} e^{p_n t}. \tag{4.35}$$

5. TAYLOR–COUETTE FLOW IN AN ANNULUS DUE TO A CONSTANT COUPLE

Consider a second grade fluid at rest in the annular region between two infinitely long co-axial cylinders. At time $t = 0^+$ *let the inner cylinder of radius* R_0 *be set in rotation about its axis by a constant torque per unit length* $2\pi R_0 f$ *and let the outer cylinder of radius* R_1 *be held stationary.*

Here, we shall apply the theorem of Section 3 to obtain the result for a fluid of second grade. However, before doing so, we first need the solution to the Navier–Stokes equation, which has not been documented to the best of our knowledge.

We shall assume the following velocity field:

$$\mathbf{v} = (0, v(r,t), 0).$$

The momentum equation reduces to

$$\mu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) + \alpha_1 \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) - \rho\frac{\partial v}{\partial t} = 0, \quad R_0 < r < R_1, \quad t > 0, \tag{5.1}$$

to be solved subject to the boundary conditions

$$v = 0, \qquad r = R_1, \quad t > 0, \tag{5.2}$$

$$\tau_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) + \alpha_1 \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) = f, \qquad r = R_0, \quad t > 0, \tag{5.3}$$

and initial condition

$$v = 0, \quad R_0 \le r \le R_1, \quad t > 0. \tag{5.4}$$

The subsidiary equation obtained after taking the Laplace transform is

$$\frac{\mathrm{d}^2 \bar{v}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\bar{v}}{\mathrm{d}r} - \left(\frac{1}{r^2} + \frac{\rho p}{\mu + \alpha_1 p}\right)\bar{v} = 0, \quad R_0 < r < R_1, \tag{5.5}$$

together with the transformed boundary conditions

$$v = 0, \quad r = R_1 \tag{5.6}$$

$$\frac{d\bar{v}}{dr} - \frac{\bar{v}}{r} = \frac{f}{p(\mu + \alpha_1 p)}, \quad r = R_0. \tag{5.7}$$

The bar denotes the Laplace transform of v. The solution of the two-point boundary value problem (5.5)–(5.7) is

$$\begin{aligned}\bar{v}(r, p) &= \frac{f}{p(\mu + \alpha_1 p)} \frac{1}{q} \frac{I_1(qr)K_1(qR_1) - I_1(qR_1)K_1(qr)}{I_2(qR_0)K_2(qR_1) + I_1(qR_1)K_2(qR_0)} \\ &= \frac{f}{\rho} \frac{q}{p^2} \frac{I_1(qr)K_1(qR_1) - I_1(qR_1)K_1(qr)}{I_2(qR_0)K_2(qR_1) + I_1(qR_1)K_2(qR_0)}\end{aligned} \tag{5.8}$$

where

$$q = \sqrt{\frac{\rho p}{\mu + \alpha_1 p}}. \tag{5.9}$$

We shall consider the principal branch of q such that $-\pi/2 < \arg q \leq \pi/2$.

For the Navier–Stokes fluid $\alpha_1 = 0$; let us define $\bar{q} = \sqrt{\rho p/\mu} = \sqrt{p/v}$. The complex inversion formula yields

$$v(r, t) = \frac{1}{2\pi i} \int_L \frac{f}{\mu p \sqrt{p/v}} \frac{I_1(\bar{q}r)K_1(\bar{q}R_1) - I_1(\bar{q}R_1)K_1(\bar{q}r)}{I_2(\bar{q}R_0)K_1(\bar{q}R_1) + I_1(\bar{q}R_1)K_2(\bar{q}R_0)} e^{pt} \, dp. \tag{5.10}$$

It is easily verified that the integrand of (5.10) is a one-to-one function of p, so to evaluate the integral (5.10) we use the contour shown in Fig. 2. The singularities of (5.10) are given by the zeros of the denominator which, after setting $p = -v\lambda^2$, becomes

$$I_2(i\lambda R_0)K_1(i\lambda R_1) + I_1(i\lambda R_1)K_2(i\lambda R_0) = \tfrac{1}{2}\pi i[J_2(\lambda R_0)Y_1(\lambda R_1) - J_1(\lambda R_1)Y_2(\lambda R_0)].$$

Here the following formulae have been used [11]:*

$$I_2(iz) = e^{\pi i}J_2(z), \quad I_1(iz) = e^{\frac{\pi}{2}i}J_1(z), \quad K_1(iz) = -\tfrac{1}{2}\pi[J_1(z) - iY_1(z)],$$
$$K_2(iz) = \tfrac{1}{2}\pi i[J_2(z) - iY_2(z)].$$

The poles of the integrand in (5.10) are $p_0 = 0$ (which is simple) and $p_n = -v\lambda_n^2$ ($n = 1, 2, \ldots$), where λ_n is the nth positive root of the transcendental equation

$$J_2(\lambda R_0)Y_1(\lambda R_1) - J_1(\lambda R_1)Y_2(\lambda R_0) = 0. \tag{5.11}$$

By the theorem given in the Appendix, all the infinitely many roots of (5.11) are real and simple. Now

$$\left[\frac{d}{d\lambda}\{J_2(\lambda R_0)Y_1(\lambda R_1) - J_1(\lambda R_1)Y_2(\lambda R_0)\}\right]_{\lambda = \lambda_1} =$$

$$R_0J_2'(\lambda_1 R_0)Y_1(\lambda_1 R_1) + R_1J_2(\lambda_1 R_0)Y_1'(\lambda_1 R_1) - R_1J_1'(\lambda_1 R_1)Y_2(\lambda_1 R_0) - R_0J_1(\lambda_1 R_1)Y_2'(\lambda_1 R_0). \tag{5.12}$$

Also, if λ_1 is a root of (5.11), then $J_2(\lambda_1 R_0)/J_1(\lambda_1 R_1) = Y_2(\lambda_1 R_0)/Y_1(\lambda_1 R_1) = k$, say, therefore

$$\begin{aligned}&\left[\frac{d}{d\lambda}\{J_2(\lambda R_0)Y_1(\lambda R_1) - J_1(\lambda R_1) - Y_2(\lambda R_0)\}\right]_{\lambda = \lambda_1} \\ &= \frac{R_0}{k}\{J_2'(\lambda_1 R_0)Y_2(\lambda_1 R_0) - J_2(\lambda_1 R_0)Y_2'(\lambda_1 R_0)\}\end{aligned}$$

Op. cit., Carslaw and Jaeger, p. 351, formulae (34) and (35).

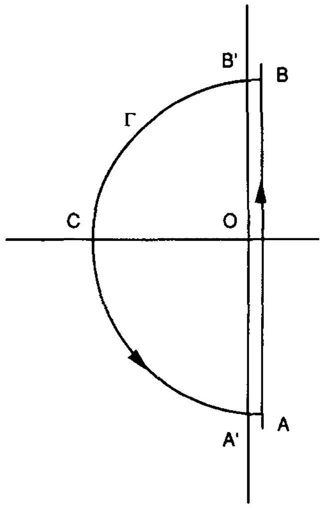

Fig. 2. Contour of integration.

$$- R_1 k\{J_1'(\lambda_1 R_1) Y_1(\lambda_1 R_1) - J_1(\lambda_1 R_1) Y_1'(\lambda_1 R_1)\} = \frac{2}{\pi\lambda_1}\left(k - \frac{1}{k}\right)$$

$$= \frac{2}{\pi\lambda_1}\left(\frac{J_2(\lambda_1 R_0)}{J_1(\lambda_1 R_1)} - \frac{J_1(\lambda_1 R_1)}{J_2(\lambda_1 R_0)}\right) = \frac{2}{\pi\lambda_1}\left(\frac{J_2^2(\lambda_1 R_0) - J_1^2(\lambda_1 R_1)}{J_1(\lambda_1 R_1) J_2(\lambda_1 R_0)}\right). \tag{5.13}$$

Thus the residue at the pole $p_n = -\nu\lambda_n^2$ $(n = 1, 2, \ldots)$ is

$$R_n = \frac{\pi f}{\mu} \frac{J_1^2(R_1\lambda_n)[J_1(\lambda_n r) Y_2(\lambda_n R_0) - Y_1(\lambda_n r) J_2(\lambda_n R_0)]}{\lambda_n [J_2^2(\lambda_n R_0) - J_1^2(\lambda_n R_1)]} \exp(-\nu\lambda_n^2 t)$$

$$= -\frac{\pi f}{\mu} \frac{J_1(\lambda_n r) Y_1(\lambda_n R_1) - J_1(\lambda_n R_1) Y_2(\lambda_n r)}{\lambda_n [J_1^2(\lambda_n R_1) - J_2^2(\lambda_n R_0)]} J_1(\lambda_n R_1) J_2(\lambda R_0) \exp(-\nu\lambda_n^2 t). \tag{5.14}$$

In the last step we have used the transcendental equation (5.11). To find the residue at the pole $p_0 = 0$, we will take advantage of the expansions of Bessel functions

$$I_1(z) = \frac{1}{2} z\left\{1 + \frac{1}{8} z^2 + \cdots\right\}, \qquad K_1(z) = \frac{1}{z} + \frac{1}{2} z \log\frac{z}{2} + \cdots,$$

$$I_2(z) = \left(\frac{1}{2} z\right)^2 (1 + \cdots), \qquad K_2(z) = \frac{1}{z^2} + \cdots,$$

valid in the neighborhood of $z = 0$ in the integrand of (5.10). Straightforward computations lead to

$$R_0 = \frac{f}{2\mu}\left(\frac{R_0}{R_1}\right)^2 \left(r - \frac{R_1^2}{r}\right). \tag{5.15}$$

It is easy to show that, ultimately:

$$v(r, t) = \frac{f}{2\mu}\left(\frac{R_0}{R_1}\right)^2 \left(r - \frac{R_1^2}{r}\right) + \frac{\pi f}{\mu} \sum_{n=1}^{\infty} \frac{J_1^2(\lambda_n R_1)[J_1(\lambda_n r) Y_2(\lambda_n R_0) - Y_1(\lambda_n r) J_2(\lambda_n R_0)]}{\lambda_n [J_2^2(\lambda_n R_0) - J_1^2(\lambda_n R_1)]} \exp(-p_n t) \tag{5.16}$$

where $p_n = \nu\lambda_n^2$. By replacing p_n with $p_n^* = \mu\lambda_n^2/(\rho + \alpha_1\lambda_n^2)$, we obtain immediately the velocity for a second grade fluid

$$v(r, t) = \frac{f}{2\mu}\left(\frac{R_0}{R_1}\right)^2 \left(r - \frac{R_1^2}{r}\right) + \frac{\pi f}{\mu} \sum_{n=1}^{\infty} \frac{J_1^2(\lambda_n R_1)[J_1(\lambda_n r) Y_2(\lambda_n R_0) - Y_1(\lambda_n r) J_2(\lambda_n R_0)]}{\lambda_n [J_2^2(\lambda_n R_0) - J_1^2(\lambda_n R_1)]} \exp(-p_n^* t). \tag{5.17}$$

6. TAYLOR–COUETTE FLOW IN AN ANNULUS

Consider a second grade fluid contained between two infinitely long coaxial cylinders of radii R_0 *and* R_1 $(R_1 > R_0)$. *The outer cylinder is held fixed while the inner one, at time* $t = 0$, *starts rotating with constant angular velocity* Ω.

Goldstein [15] uses the Laplace transform to attack this problem in the case of a viscous fluid when both the cylinders at time $t = 0$ start rotating with different constant angular velocities. Ramkissoon [16] considers a second grade fluid and assumes that the angular velocity of the inner cylinder obeys the generic law $\Omega = \Omega(t)$. He employs the Hankel and the Laplace transform, but his solution is not completely correct. In fact, let us recall the Laplace transform of $\mathrm{d}\Omega(t)/\mathrm{d}t$

$$L\left\{\frac{\mathrm{d}\Omega(t)}{\mathrm{d}t}\right\} = pL\{\Omega(t)\} - \Omega(0^+).$$

Ramkissoon ignores the term $\Omega(0^+)$ (cf. [16], equation (18)) and, therefore, his solution does not meet the initial condition. Later on, the particular case that he studies, that is, $\Omega(t) = \sin t$, yields the correct solution by chance, for $\Omega(0^+)$ is zero in this case.

Here, the method of separation of variables will be employed to solve this problem. The assumed form for the velocity field is

$$\mathbf{v} = (0, v(r, t), 0).$$

The governing equation is, therefore

$$\mu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) + \alpha_1 \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) - \rho\frac{\partial v}{\partial t} = 0, \quad R_0 < r < R_1, \quad t > 0 \tag{6.1}$$

with boundary conditions

$$v = \Omega R_0, \quad r = R_0, \quad t > 0, \tag{6.2}$$

$$v = 0, \quad r = R_1, \quad t > 0, \tag{6.3}$$

and initial condition

$$v = 0, \quad R_0 \le r \le R_1, \quad t = 0. \tag{6.4}$$

It is easy to show that the generic solution to equation (6.1) has the form

$$v = R(r)\exp\left(-\frac{\mu\lambda^2}{\rho + \alpha_1\lambda^2}t\right), \tag{6.5}$$

where $R(r)$ and λ represent, respectively, the eigenfunctions and eigenvalues of the eigenvalue problem (cf. [17])*

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{\mathrm{d}R}{\mathrm{d}r} + \left(\lambda^2 - \frac{1}{r^2}\right)R = 0, \tag{6.6}$$

$$R = \Omega R_0, \quad r = R_0, \tag{6.7}$$

$$R = 0, \quad r = R_1. \tag{6.8}$$

The complete solution is

$$v(r, t) = \frac{\Omega R_1^2}{r}\frac{r^2 - R_0^2}{R_1^2 - R_0^2} - \pi\Omega R_1 \sum_{n=1}^{\infty} \frac{J_1(R_0\lambda_n)Y_1(r\lambda_n) - Y_1(R_0\lambda_n)J_1(r\lambda_n)}{J_1^2(R_1\lambda_n) - J_1^2(R_0\lambda_n)}\exp\left(-\frac{\mu\lambda_n^2}{\rho + \alpha_1\lambda_n^2}t\right), \tag{6.9}$$

where λ_n is the nth root of the transcendental equation

$$J_1(R_0\lambda)Y_1(R_1\lambda) - J_1(R_1\lambda)Y_1(R_0\lambda) = 0. \tag{6.10}$$

**Op. cit.*, Osizik, p. 113.

All roots of (6.10) are known to be real, simple, and infinitely many (cf. [15]). We notice that the result expressed by (6.10) could be reached simply by applying the theorem stated in Section 3 and using the result for the Navier–Stokes fluid available in the literature [11].*

7. AXIAL COUETTE FLOW IN AN ANNULUS

Consider a second grade fluid at rest in the annular region between two infinitely long coaxial cylinders. At time $t = 0$ *let the inner cylinder of radius* R_0 *start translating along its axis of symmetry with constant velocity* W *and the outer one of radius* R_1 *be held fixed.*

With the assumption that the velocity field has the form

$$\mathbf{w} = (0, 0, w(r, t))$$

the balance of linear momentum yields

$$\mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \alpha_1 \frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \rho\frac{\partial w}{\partial t} = 0, \quad R_0 < r < R_1, \quad t > 0 \tag{7.1}$$

with boundary conditions

$$w = W, \quad r = R_0, \quad t > 0, \tag{7.2}$$

$$w = 0, \quad r = R_1, \quad t > 0, \tag{7.3}$$

and initial condition

$$w = 0, \quad R_0 \le r \le R_1, \quad t > 0. \tag{7.4}$$

The mathematical formulation of this problem for a viscous fluid is identical to that of diffusion of temperature in a solid bounded by two cylindrical surfaces† (cf. [18]). By appealing to the theorem in Section 3, the solution is readily obtained as

$$\frac{w(r,t)}{W} = \frac{1}{\ln m}\left(\ln\frac{R_1}{r}\right) + \pi\sum_{n=1}^{\infty}\left\{V_0\left(\mu_n\frac{r}{R_1}\right)\frac{J_0^2(m\mu_n)}{J_0^2(\mu_n) - J_0^2(m\mu_n)}\exp\left(-\frac{\mu\lambda_n^2}{\rho + \alpha_1\lambda_n^2}t\right)\right\}, \tag{7.5}$$

where $\mu_n = \lambda_n R_0$, $m = R_1/R_0$, and

$$V_0(\lambda_n r) = J_0(\lambda_n r)Y_0(\lambda_n R_0) - J_0(\lambda_n R_0)Y_0(\lambda_n r)$$

and λ_n are the positive eigenvalues (cf. [19])‡ (all simple and real) of the transcendental equation

$$J_0(\lambda R_1)Y_0(\lambda R_0) - J_0(\lambda R_0)Y_0(\lambda R_1) = 0. \tag{7.6}$$

8. FLOW DUE TO AN IMPULSIVE PRESSURE GRADIENT OR BODY FORCE

Consider an impulsive pressure gradient or body force applied at time $t = 0^+$ *to a second grade fluid contained between two infinite parallel plates at distance* $2h$ *apart.*

Impulsive functions represent convenient tools in the idealization of certain types of problems of practical importance in physics and engineering. *In this case the theorem of Section 3 cannot be applied* because the pressure gradient experiences an impulsive and not a step change in time.

We shall assume the following velocity field

$$\mathbf{v} = (0, 0, v(x, t)).$$

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* *Op. cit.*, Carslaw and Jaeger, p. 172.

† *Op. cit.*, Luikov, p. 156.

‡ *Op. cit.*, Gray and Matthews.

The balance of linear momentum yields:

$$\rho \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial x^2} + \alpha_1 \frac{\partial^3 v}{\partial t\, \partial x^2} + k\delta(t), \quad -h < x < h, \quad t > 0, \tag{8.1}$$

with boundary conditions

$$v = 0, \quad x = \pm h, \quad t > 0, \tag{8.2}$$

and initial condition

$$v = 0, \quad -h \le x \le h, \quad t = 0, \tag{8.3}$$

where

$$k\delta(t) = \frac{\partial p}{\partial y} + \rho b,$$

$\delta(t)$ being the Dirac measure.

Notice that the mixed initial-boundary value problem (8.1)–(8.3) is equivalent, by symmetry, to the following:

$$\rho \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial x^2} + \alpha_1 \frac{\partial^3 v}{\partial t\, \partial x^2} + k\delta(t), \quad -h < x < h, \quad t > 0, \tag{8.4}$$

$$\frac{\partial v}{\partial x} = 0, \quad x = 0, \quad t > 0, \tag{8.5}$$

$$v = 0, \quad x = h, \quad t > 0, \tag{8.6}$$

$$v = 0, \quad -h \le x \le h, \quad t = 0. \tag{8.7}$$

Let us multiply both sides of (8.4) by $\cos \beta_n x$, β_n being the positive roots of the equation $\cos \beta h = 0$ (i.e. $\beta_n = (2n-1)\pi/2h$, $n = 1, 2, \ldots$). Provided that $v(x, t)$ satisfies the Dirichlet conditions over the interval $[0, h]$, we find, after applying the boundary conditions (8.5) and (8.6), that

$$\rho \frac{\mathrm{d}\bar{v}(n,t)}{\mathrm{d}t} = -\beta_n^2 \left[\mu \bar{v}(n,t) + \alpha_1 \frac{\mathrm{d}\bar{v}(n,t)}{\mathrm{d}t} \right] + \bar{k}(n)\delta(t), \tag{8.8}$$

where

$$\bar{k}(n) \doteq \int_0^h k \cos \beta_n x \, \mathrm{d}x = (-1)^{n+1} \frac{k}{\beta_n}, \tag{8.9}$$

and

$$\bar{v}(n,t) \doteq \int_0^h v(x,t) \cos \beta_n x \, \mathrm{d}x, \tag{8.10}$$

are defined as the Fourier cosine transform of k and $v(x, t)$, respectively. Now

$$v(t,x) = \sum_{n=1}^{\infty} \frac{\bar{v}(n,t)}{N(\beta_n)} \cos \beta_n x, \tag{8.11}$$

is, by definition, the inverse Fourier cosine transform of $v(t, x)$, where $1/N(\beta_n) = 2/h$.

The time variable is eliminated by taking the Laplace transform of (8.8) and applying the initial condition (8.7):

$$(\rho + \alpha_1 \beta_n^2) p L\{\bar{v}(n,t)\} + \mu \beta_n^2 L\{\bar{v}(n,t)\} = (-1)^{n+1} \frac{k}{\beta_n}. \tag{8.12}$$

Hence

$$L\{\bar{v}(n,t)\} = \frac{(-1)^{n+1}}{\beta_n} \frac{k}{(\rho + \alpha_1 \beta_n^2) p + \mu \beta_n^2} = \frac{(-1)^{n+1}}{(\rho + \alpha_1 \beta_n^2)\beta_n} \frac{k}{p + \dfrac{\mu \beta_n^2}{\rho + \alpha_1 \beta_n^2}}. \tag{8.13}$$

The inverse Laplace transform of (8.13) is

$$\bar{v}(n,t)=\frac{(-1)^{n+1}k}{(\rho+\alpha_1\beta_n^2)\beta_n}\exp\left\{-\frac{\mu\beta_n^2}{(\rho+\alpha_1\beta_n^2)}t\right\}, \tag{8.14}$$

whose inverse Fourier cosine transform by virtue of (8.11) leads to

$$v(x,t)=\frac{2k}{h}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(\rho+\alpha_1\beta_n^2)\beta_n}\cos\beta_n x\exp\left\{-\frac{\mu\beta_n^2}{(\rho+\alpha_1\beta_n^2)}t\right\}. \tag{8.15}$$

Now let us study the solution (8.15) at $t=0$:

$$v(x,0)=\frac{2k}{h}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(\rho+\alpha_1\beta_n^2)\beta_n}\cos\beta_n x. \tag{8.16}$$

If $\alpha_1=0$, we recover the result for the Newtonian fluid:

$$v(x,0)=\frac{2k}{\rho h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\beta_n x}{\beta_n}. \tag{8.17}$$

Recall that [17]*

$$\frac{2}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\beta_n x}{\beta_n}=\left|\begin{matrix}0 & \text{at } x=h\\ 1 & \text{at } x\neq h\end{matrix}\right. .$$

Thus, the initial condition (fluid at rest at $t=0$) is not satisfied! This is not surprising: a similar example of velocity discontinuity at time $t=0$ arises in the problem of a block of mass m subject to a blow P [11].† However, it is puzzling that the incipient velocity profile is flat and presents a discontinuity at the wall.

For a second grade fluid, after recalling that [17]‡

$$\frac{2}{h}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos\beta_n x}{\beta_n^3}=-\frac{1}{2}(x^2-h^2), \tag{8.18}$$

and supposing that $\alpha_1\beta_n^2\gg\rho$, the velocity profile at $t=0^+$ can be approximated as

$$v(x,0)\cong\frac{k}{2\alpha_1}(h^2-x^2). \tag{8.19}$$

Therefore, the velocity profile in the impending motion of a second grade fluid is physically sound and resembles the steady-state one, where the normal stress modulus α_1 replaces the viscosity μ. Although it does not meet the initial condition, the velocity is infinitely differentiable in the spacial variable and seems better behaved than a Navier–Stokes fluid. This is true in other instances; Ting [7], for example, pointed out that the second grade fluid does not give rise to the paradox noticed by Lamb [20] regarding the rate of increase of the rate of dissipation shown by a viscous fluid. In order for the velocity to die out, as physically expected, the stress moduli α_1 must be positive, otherwise the flow would be unstable.

9. COUETTE FLOW FORMATION BETWEEN PARALLEL PLATES

Consider a second grade fluid at rest between two infinite parallel plates at a distance 2h *apart. At time* t $=0^+$ *suppose that the lower wall moves with constant velocity in a direction parallel to the upper one which is stationary.*

The solution for the classical viscous fluid is well known and available in the literature [21]. We shall assume the following velocity field

$$\mathbf{v}=(0,0,v(x,t)).$$

* *Op. cit.*, Osizik, p. 517.
† *Op. cit.*, Carslaw and Jaeger, p. 234.
‡ *Op. cit.*, Osizik, p. 517.

The balance of linear momentum leads to

$$\rho\frac{\partial v}{\partial t}=\mu\frac{\partial^2 v}{\partial x^2}+\alpha_1\frac{\partial^3 v}{\partial t\,\partial x^2},\quad 0<x<h,\quad t>0, \tag{9.1}$$

with boundary conditions

$$v=0,\quad x=h,\quad t>0, \tag{9.2}$$

$$v=U,\quad x=0,\quad t>0, \tag{9.3}$$

and initial condition

$$v=0,\quad 0\le x\le h,\quad t=0. \tag{9.4}$$

The method of separation of variables yields a solution (cf. [10]) in the form of a series of products of trigonometric and exponential functions:

$$v(x,t)=U\left(1-\frac{x}{h}\right)-\frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{n^2\pi^2\beta t}{h^2}\right)\sin\frac{n\pi x}{h}, \tag{9.5}$$

where

$$\beta=\frac{\mu}{\rho+\alpha_1 n^2\dfrac{\pi^2}{h^2}}.$$

For large values of time this series converges very rapidly, but for small times the solution in this form is not useful for computational purposes. Therefore, we are going to provide an alternative form using the method of the Laplace transform.

Let us define the following dimensionless variables

$$\tau\doteq\left(\frac{\mu}{\alpha_1}\right)t, \tag{9.6}$$

$$\xi\doteq\left(\frac{\rho}{\alpha_1}\right)^{1/2}x. \tag{9.7}$$

Let us also define

$$\xi_0\doteq\left(\frac{\rho}{\alpha_1}\right)^{1/2}h. \tag{9.8}$$

Notice at once that the Navier–Stokes solution cannot be retrieved from that for fluids of second grade by letting $\alpha_1\to 0$ in virtue of the transformations (9.6) and (9.8).

It follows from (9.1), (9.6) and (9.7) that

$$\partial_\xi^2 V+\partial_\tau\partial_\xi^2 V=\partial_\tau V, \tag{9.9}$$

where the transcript denotes partial differentiation with respect to that variable.

In deriving (9.9), we have assumed that

$$v\doteq\frac{\mathrm{V}}{\mathrm{U}}, \tag{9.10}$$

where U is some characteristic speed, which in the present case would be the speed imposed on the plate at $x=0$ at time $t=0^+$.

Since the fluid is at rest for all $\tau\le 0$,

$$V=0,\quad 0\le\xi\le\xi_0,\quad \tau=0. \tag{9.11}$$

Also, as the lower plate is maintained at a constant speed U for all $\tau>0$, we have

$$V=1,\quad \xi=0,\quad \tau>0, \tag{9.12}$$

whereas for the upper plate

$$V=0,\quad \xi=\xi_0,\quad \tau>0. \tag{9.13}$$

Applying the Laplace transform

$$L\{f(\xi,\tau)\} \doteq \int_0^\infty e^{-p\tau} f(\xi,\tau)\,d\tau,$$

to (9.9) and using (9.11), we obtain

$$pL\{V(\xi,\tau)\} = (1+p)L\{V(\xi,\tau)_{\xi\xi}. \tag{9.14}$$

The suffix ξ denotes partial derivative with respect to ξ. Since

$$L\{V(0,\tau)\} = \frac{1}{p}, \tag{9.15}$$

$$L\{V(\xi_0,\tau)\} = 0, \tag{9.16}$$

it follows from (9.14)–(9.16) that

$$L\{V(\xi,\tau)\} = \frac{1}{p}\left(e^{-m\xi} + \frac{e^{m\xi} - e^{-m\xi}}{1 - e^{2m\xi_0}}\right). \tag{9.17}$$

Here $m = (p/(p+1))^{1/2}$, we choose the branch of m for which

$$m = \varphi + i\psi, \quad -\frac{\pi}{2} < \arg m \le \frac{\pi}{2}.$$

This implies that $\varphi \ge 0$. Another form of (9.17) is

$$L\{v(\xi,\tau)\} = \frac{1}{p}\left[\quad e^{-m\xi} - e^{-m\xi_0}\left(\frac{e^{m\xi} - e^{-m\xi}}{e^{m\xi_0} - e^{-m\xi_0}}\right)\right] \quad = \frac{1}{p}\frac{\sinh m(\xi_0 - \xi)}{\sinh m\xi_0}. \tag{9.18}$$

Consider the term

$$\frac{1}{p}\frac{e^{m\xi} - e^{-m\xi}}{1 - e^{2m\xi_0}} = -\frac{1}{p}\frac{1}{e^{2m\xi_0}}\frac{e^{m\xi} - e^{-m\xi}}{1 - e^{-2m\xi_0}}, \tag{9.19}$$

in (9.17) and recall that

$$\frac{1}{1 - e^{-2m\xi_0}} = \sum_{i=0}^{\infty} e^{-2im\xi_0}. \tag{9.20}$$

The sum of the geometric progression (9.20) converges because $|e^{-2m\xi}| < 1$; in fact

$$|e^{-2m\xi}| = |\exp(-2(\varphi + i\psi)\xi)| = \exp(-2\varphi\xi) \le 1.$$

The equal sign is valid when $\varphi \equiv 0$, which occurs if and only if $p \equiv 0$ which is excluded because it is a singularity for $L\{V(\xi,\tau)\}$. From (9.20) we have

$$\frac{1}{e^{2m\xi_0}}\frac{1}{1 - e^{-2m\xi_0}} = \sum_{i=0}^{\infty} e^{-2m\xi_0(1+i)} = \sum_{i=1}^{\infty} e^{-2im\xi_0}, \tag{9.21}$$

and, by virtue of (9.19) and (9.21), formula (9.17) can be rewritten as

$$L\{V(\xi,\tau)\} = \frac{1}{p}e^{-m\xi} + \frac{1}{p}\sum_{i=0}^{\infty}\{\exp[-m(2i\xi_0 + \xi)] - \exp[-m(2i\xi_0 - \xi)]\}. \tag{9.22}$$

It has been shown (cf. [21]) that

$$\frac{1}{p}e^{-mq} = L\left\{\exp(-\tau)\int_0^\infty \exp(-\eta) I_0(2\eta^{1/2}\tau^{1/2})\,\mathrm{erfc}\left(\frac{1}{2}q\eta^{1/2}\right)d\eta\right\}. \tag{9.23}$$

It follows from (9.22) and (9.23) that

$$V(\xi,\tau)=\exp(-\tau)\int_0^\infty \exp(-\eta)I_0(2\eta^{1/2}\tau^{1/2})\left\{\operatorname{erfc}\left(\frac{1}{2}\xi\eta^{-1/2}\right)\right.$$
$$\left.-\sum_{i=1}^{\infty}\left\{\operatorname{erfc}\left[\left(i\xi_0-\frac{\xi}{2}\right)\eta^{-1/2}\right]-\operatorname{erfc}\left[\left(i\xi_0+\frac{\xi}{2}\right)\eta^{-1/2}\right]\right\}\right\}d\eta, \tag{9.24}$$

and, in dimensional form

$$v(x,t)=U\exp\left(-\frac{\mu}{\alpha_1}t\right)\int_0^\infty \exp(-\eta)I_0\left[2\left(\frac{\mu\eta}{\alpha_1}\right)^{1/2}t^{1/2}\right]\left\{\operatorname{erfc}\left[\frac{1}{2}\left(\frac{\rho}{\alpha_1}\right)^{1/2}x\eta^{-1/2}\right]\right.$$
$$\left.-\sum_{i=1}^{\infty}\left\{\operatorname{erfc}\left[\left(\frac{\rho}{\alpha_1}\right)^{1/2}\left(ih-\frac{x}{2}\right)\eta^{-1/2}\right]-\operatorname{erfc}\left[\left(\frac{\rho}{\alpha_1}\right)^{1/2}\left(ih+\frac{x}{2}\right)\eta^{-1/2}\right]\right\}\right\}d\eta. \tag{9.25}$$

The solution can be cast in a slightly different form if we express (9.18) as

$$L\{V(\xi,\tau)\}=\frac{1}{p}\frac{\sinh m(\xi_0-\xi)}{\sinh m\xi_0}=\frac{1}{p}\frac{e^{m(\xi_0-\xi)}-e^{-m(\xi_0-\xi)}}{1-e^{-2m\xi_0}}e^{-m\xi_0}$$
$$=\frac{e^{m(\xi_0-\xi)}-e^{-m(\xi_0-\xi)}}{p}\sum_{n=0}^{\infty}e^{-(2n+1)m\xi_0}$$
$$=\frac{1}{p}\sum_{n=0}^{\infty}\{\exp\{-m[(2n+1)\xi_0-(\xi_0-\xi)]\}-\exp\{-m[(2n+1)\xi_0+(\xi_0-\xi)]\}\}$$
$$=\frac{1}{p}\sum_{n=0}^{\infty}\{\exp[-m(2n\xi_0+\xi)]-\exp[-m((2n+1)\xi_0-\xi)]\}. \tag{9.26}$$

Using (9.23), we obtain

$$V(\xi,\tau)=\exp(-\tau)\int_0^\infty \exp(-\eta)I_0(2\eta^{1/2}\tau^{1/2})\sum_{n=0}^{\infty}\left\{\operatorname{erfc}\left[\frac{1}{2}(2n\xi_0+\xi)\eta^{-1/2}\right]\right.$$
$$\left.-\left\{\operatorname{erfc}\left[\frac{1}{2}(2(n+1)\xi_0-\xi)\eta^{-1/2}\right]\right\}\right\}d\eta, \tag{9.27}$$

and, in dimensional form:

$$v(x,t)=U\exp\left(-\frac{\mu}{\alpha_1}t\right)\int_0^\infty \exp(-\eta)I_0\left(2\left(\frac{\mu\eta}{\alpha_1}\right)^{1/2}t^{1/2}\right)$$
$$\cdot\sum_{n=0}^{\infty}\left\{\operatorname{erfc}\left[\frac{1}{2}(2nh+x)\left(\frac{\rho}{\alpha_1}\right)^{1/2}\eta^{-1/2}\right]\right.$$
$$\left.-\operatorname{erfc}\left[\frac{1}{2}(2(n+1)h-x)\left(\frac{\rho}{\alpha_1}\right)^{1/2}\eta^{1/2}\right]\right\}. \tag{9.28}$$

It is easy to show that both (9.25) and (9.28) satisfy (9.1) and the boundary conditions but not the initial condition. Therefore, they do not represent smooth solutions (cf. [22]); none the less they are physically interesting, for each term of the series corresponds to the solution of a related problem for a semi-infinite region. Hence the solution for the finite region can be regarded as the superposition of successively reflected waves [11].*

**Op. cit.*, Carslaw and Jaeger, p. 174.

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APPENDIX

Theorem

If n is an integer and a, b are real and positive, then the function

$$g(z) = J_n(az)\,Y_{n+1}(bz) - J_{n+1}(bz)\,Y_n(az), \tag{A1}$$

is odd, continuous, except at $z = 0$, and has infinitely many zeros which are all simple and real.

Proof

Without loss of generality, we will assume $b > a$.

(a) Let us show that the function $g(z)$ is odd.

Recall that [11]*

$$Y_n(-z) = Y_n(ze^{\pi i}) = e^{-n\pi i}Y_n(z) + 2i\cos n\pi J_n(z), \tag{A2}$$

and

$$J_n(-z) = J_n(ze^{\pi i}) = e^{-n\pi i}J_n(z). \tag{A3}$$

Substituting these expressions in (A1), we get

$$g(-z) = e^{-n\pi i}J_n(az)[e^{-(n+1)\pi i}Y_{n+1}(bz) + 2i\cos(n+1)\pi J_{n+1}(bz)]$$
$$- e^{-(n+1)\pi i}J_{n+1}(bz)[e^{-n\pi i}Y_n(az) + 2i\cos n\pi J_n(az)].$$

After simplification, we find

$$g(-z) = -e^{-2n\pi i}g(z). \tag{A4}$$

Therefore, in general, $g(z)$ is an odd function except for a multiplicative constant.

(b) We will prove in this section that $g(z)$ does not possess any purely imaginary zero.

Let us write $g(z)$ as

$$J_n(az)J_{n+1}(bz)\left(\frac{Y_{n+1}(bz)}{J_{n+1}(bz)} - \frac{Y_n(az)}{J_n(az)}\right), \tag{A5}$$

Op. cit., Carslaw and Jaeger, p. 351.

provided that $J_n(az)$ and $J_{n+1}(bz)$ do not vanish. Recall that [14]* $J_\nu(ze^{\pi i}) = e^{\nu\pi i}J_\nu(z)$ (ν is regarded as a complex number); hence

$$J_\nu(aze^{\pi i})J_{\nu+1}(bze^{\pi i}) = e^{(\nu+\nu+1)\pi i}J_\nu(az)J_{\nu+1}(bz) = -e^{2\nu\pi i}J_\nu(az)J_{\nu+1}(bz).$$

This means that $J_n(az)J_{n+1}(bz)$ is an odd function except for a multiplicative constant. Therefore, the function $(Y_{n+1}(bz)/J_{n+1}(bz)) - (Y_n(az)/J_n(az))$ is even except for a multiplicative constant. Define the function

$$f(z) = \frac{Y_{n+1}(bz)}{J_{n+1}(bz)} - \frac{Y_n(az)}{J_n(az)}. \tag{A6}$$

It can be shown that [24]† $f(ix)$, with x real, is real. Now

$$\frac{d}{dx}\frac{Y_n(ix)}{J_n(ix)} = \frac{2}{\pi x J_n^2(ix)} = \frac{2}{\pi x(i^n I_n(x))} = (-1)^n\frac{2}{\pi x I_n^2(x)}. \tag{A7}$$

Hence

$$f'(ix) = (-1)^{n+1}\frac{2}{\pi x}\left(\frac{1}{I_{n+1}^2(ax)} + \frac{1}{I_n^2(bx)}\right). \tag{A8}$$

It is clear from (A8) that $f(ix)$ and $f'(ix)$ are monotone and that $\lim_{x\to\infty}f'(ix) = 0$. Let us investigate the asymptotic behavior of $f(ix)$. When $|z|$ is small [13]‡

$$\frac{Y_p(p)}{J_p(z)} \cong \frac{\dfrac{-2^p(p-1)!z^{-p}}{\pi}}{\dfrac{1}{2^p p!z^p}} = -\frac{2^{2p}}{\pi}p!(p-1)!z^{-2p}. \tag{A9}$$

Thus for small values of x

$$f(ix) = \frac{Y_{n+1}(ibx)}{J_{n+1}(ibx)} - \frac{Y_n(iax)}{J_n(iax)} \cong -\frac{2^{2(n+1)}n!(n+1)!}{\pi(ibx)^{2(n+1)}} - \frac{-2^{2n}n!(n-1)!}{\pi(iax)^{2n}}$$

$$= (-1)^n\left(\frac{k_1}{(bx)^{2(n+1)}} + \frac{k_2}{(ax)^{2n}}\right)$$

(k_1 and k_2 are constants) and

$$\lim_{x\to 0} f(ix) = \infty.$$

Let us study the case when x goes to infinity. Consider the identity§ (cf. [23])

$$J_n(z)Y_{n+1}(z) - J_{n+1}(z)Y_n(z) = -\frac{2}{\pi z}, \tag{A10}$$

which, provided that z is not a zero of $J_n(z)$ or $J_{n+1}(z)$ (we omit the proof in this case, but it follows along similar lines), is equivalent to

$$\frac{Y_{n+1}(z)}{J_{n+1}(z)} = \frac{Y_n(z)}{J_n(z)} - \frac{2}{\pi z J_{n+1}(z)J_n(z)}. \tag{A11}$$

By virtue of (A11), $f(ix)$ can be written as

$$f(ix) = \frac{Y_n(ibx)}{J_n(ibx)} - \frac{Y_n(iax)}{J_n(iax)} - \frac{2(-1)^n}{\pi bx I_{n+1}(bx)I_n(bx)} = \frac{Y_n(ibx)}{i^n I_n(bx)} - \frac{Y_n(iax)}{i^n I_n(ax)} - \frac{2(-1)^n}{\pi bx I_{n+1}(bx)I_n(bx)}. \tag{A12}$$

Observe that the real function $f(ix)$ is continuous on the whole real domain $(-\infty, +\infty)$ except at $x = 0$. From (A12)

$$\lim_{x\to\infty} f(ix) = \lim_{x\to\infty}\left(\frac{Y_n(ibx)}{J_n(ibx)} - \frac{Y_n(iax)}{J_n(iax)} - \frac{2(-1)^n}{\pi bx I_{n+1}(bx)I_n(bx)}\right) = \lim_{x\to\infty}\left(\frac{Y_n(ibx)}{J_n(ibx)} - \frac{Y_n(iax)}{J_n(iax)}\right). \tag{A13}$$

Now, recall¶ that (cf. [24])

$$y(x) = \frac{Y_n(ix)}{J_n(ix)} - i,$$

Op. cit., MacLachlan, p. 191.
†*Op. cit.*, Carslaw, p. 128.
‡*Op. cit.*, Hildebrand, p. 148.
§*Op. cit.*, Watson, p. 77, formula (12).
¶*Op. cit.*, Carslaw, p. 128.

is real. Also

$$y'(x) = (-1)^n \frac{2}{\pi x I_n^2(x)}.$$

If n is even (parallel considerations apply if n is odd), then $y'(x) > 0$; thus $y(x)$ is monotone and (since $b > a$)

$$y(bx) - y(ax) \geq 0, \tag{A14}$$

which is also valid when $x \to \infty$. The only case for (A14) to be an equality is when $x \to \infty$ because $y'(bx) - y'(ax)$ is always negative and vanishes only at infinity [if n is odd, $y'(x) < 0 \to y(bx) - y(ax) \leq 0$]. We can then confidently claim that

$$\lim_{x \to \infty} f(ix) > 0.$$

Let us summarize the properties of $f(ix)$ when x is real (we will consider the case of n even; similar arguments apply when n is odd):

(1) it is an even real function analytic in the whole real domain $(-\infty, +\infty)$ with its derivatives except at $x = 0$;
(2) $\lim_{x \to \infty} f(ix) = +\infty$, $\lim_{x \to \infty} f(ix) = c > 0$;
(3) $\text{signum}[f'(ix)] = -\text{signum}[x]$, thus $f(ix)$ continually increases $\forall x < 0$ and decreases $\forall x > 0$;
(4) $f'(ix) \neq 0\ \forall x$, but $\lim_{x \to \infty} f'(ix) = 0$.

Moreover, the Lagrange theorem ensures that $f''(ix) = k > 0$.

We conclude that $f(ix)$ is strictly positive for $\forall x \in R$. In fact, property (2) assures the existence of a point x_∞ such that $\forall |x| \geq |x_\infty|$, $f(x) \geq 0$. Let us call x_1 a point for which $|x_1| \geq |x_\infty|$. By applying the Lagrangian theorem

$$\frac{f(ix) - f(ix_1)}{x - x_1} = f'(i\xi), \tag{A15}$$

where

$$0 < x \leq \xi \leq x_1 \quad \text{or} \quad x_1 \leq \xi \leq x < 0.$$

It immediately follows from property (3) and from (A15) that

$$f(ix) = f'(i\xi)(x - x_1) + f(ix_1) > 0. \tag{A16}$$

Therefore, $f(ix)$ never vanishes and $g(z)$ cannot have imaginary zeros.

(c) The function $g(x)$ does not have any repeated zero.

In order for a root $x = \alpha$ to be multiple, a necessary condition is that $g(\alpha) = g'(\alpha) = 0$. Now, define the following functions of ρ

$$u = J_n(ax) Y_{n+1}(\rho x) - J_{n+1}(\rho x) Y_n(ax),$$

$$v = J_n(ay) Y_{n+1}(\rho y) - J_{n+1}(\rho y) Y_n(ay).$$

From the Lommel integral [19]*

$$\int_a^b u^2 \rho \, d\rho = -\frac{1}{2x} \left\{ \rho \left(u \frac{\partial^2 u}{\partial \rho \partial x} - \frac{\partial u}{\partial x} \frac{\partial u}{\partial \rho} \right) \right\}_{\rho = b}, \tag{A17}$$

we see that, if $x = \alpha$, the right-hand side vanishes but the left-hand side is positive: thus, all the zeros must be simple.

(d) Let us show that $g(z)$ has no complex zeros. Observe that $Y_n(z) = \bar{Y}_n(z)$; this can be deduced from the definition of $Y_n(z)$ [14]†

$$Y_n(z) = \lim_{\nu \to n} \frac{J_\nu(z) \cos \nu\pi - J_{-\nu}(z)}{\sin \nu\pi},$$

since $J_n(\bar{z}) = \bar{J}_n(z)$; hence $g(\bar{z}) = \bar{g}(z)$. The Lommel integral

$$\int_a^b uv\rho \, d\rho = \frac{1}{x^2 - y^2} \left\{ \rho \left(u \frac{dv}{d\rho} - v \frac{du}{d\rho} \right) \right\}_{\rho = b}, \tag{A18}$$

helps us to understand that $g(z)$ cannot have any complex roots: in fact, suppose that in (A18) $x = \lambda$, $y = \bar{\lambda}$ are two complex conjugate roots of $g(z)$. Then the right-hand side of (A18) vanishes, whereas the left-hand side remains positive, for $v(\bar{\lambda}) = \bar{u}(\lambda)$.

It may be inferred from the asymptotic expansion of J_n and Y_n that $g(z)$ has infinitely many roots and that, if λ_n is the nth root, then

$$\frac{\lambda_n}{n} = O(1) \quad \text{as } n \to 0.$$

Op. cit., Gray and Matthews, p. 70, formulae (32) and (33).

† *Op. cit.*, McLachlan, p. 196, formula (107).