

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise A, Question 1

#### Question:

Prove by the method of mathematical induction, the following statement for  $n \in \mathbb{Z}^+$ .

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1)$$

#### Solution:

$$\begin{aligned} n = 1; \text{LHS} &= \sum_{r=1}^1 r = 1 \\ \text{RHS} &= \frac{1}{2}(1)(2) = 1 \end{aligned}$$

As LHS = RHS, the summation formula is true for  $n = 1$ .

Assume that the summation formula is true for  $n = k$ .

$$\text{ie. } \sum_{r=1}^k r = \frac{1}{2}k(k+1).$$

With  $n = k + 1$  terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{k+1} r &= 1 + 2 + 3 + \dots + k + (k+1) \\ &= \frac{1}{2}k(k+1) + (k+1) \\ &= \frac{1}{2}(k+1)(k+2) \\ &= \frac{1}{2}(k+1)(k+1+1) \end{aligned}$$

Therefore, summation formula is true when  $n = k + 1$ .

If the summation formula is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the result is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

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## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise A, Question 2

#### Question:

Prove by the method of mathematical induction, the following statement for  $n \in \mathbb{Z}^+$ .

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

#### Solution:

$$\begin{aligned} n = 1; \text{LHS} &= \sum_{r=1}^1 r^3 = 1 \\ \text{RHS} &= \frac{1}{4}(1)^2(2)^2 = \frac{1}{4}(4) = 1 \end{aligned}$$

As LHS = RHS, the summation formula is true for  $n = 1$ .

Assume that the summation formula is true for  $n = k$ .

$$\text{ie. } \sum_{r=1}^k r^3 = \frac{1}{4}k^2(k+1)^2.$$

With  $n = k + 1$  terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{k+1} r^3 &= 1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 \\ &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \frac{1}{4}(k+1)^2[k^2 + 4(k+1)] \\ &= \frac{1}{4}(k+1)^2(k^2 + 4k + 4) \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \\ &= \frac{1}{4}(k+1)^2(k+1+1)^2 \end{aligned}$$

Therefore, summation formula is true when  $n = k + 1$ .

If the summation formula is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the result is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

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## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise A, Question 3

#### Question:

Prove by the method of mathematical induction, the following statement for  $n \in \mathbb{Z}^+$ .

$$\sum_{r=1}^n r(r-1) = \frac{1}{3}n(n+1)(n-1)$$

#### Solution:

$$n = 1; \text{LHS} = \sum_{r=1}^1 r(r-1) = 1(0) = 0$$

$$\text{RHS} = \frac{1}{3}(1)(2)(0) = 0$$

As LHS = RHS, the summation formula is true for  $n = 1$ .

Assume that the summation formula is true for  $n = k$ .

$$\text{ie. } \sum_{r=1}^k r(r-1) = \frac{1}{3}k(k+1)(k-1).$$

With  $n = k + 1$  terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{k+1} r(r-1) &= 1(0) + 2(1) + 3(2) + \dots + k(k-1) + (k+1)k \\ &= \frac{1}{3}k(k+1)(k-1) + (k+1)k \\ &= \frac{1}{3}k(k+1)[(k-1) + 3] \\ &= \frac{1}{3}k(k+1)(k+2) \\ &= \frac{1}{3}(k+1)(k+2)k \\ &= \frac{1}{3}(k+1)(k+1+1)(k+1-1) \end{aligned}$$

Therefore, summation formula is true when  $n = k + 1$ .

If the summation formula is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the result is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

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## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise A, Question 4

#### Question:

Prove by the method of mathematical induction, the following statement for  $n \in \mathbb{Z}^+$ .

$$(1 \times 6) + (2 \times 7) + (3 \times 8) + \dots + n(n+5) = \frac{1}{3}n(n+1)(n+8)$$

#### Solution:

The identity  $(1 \times 6) + (2 \times 7) + (3 \times 8) + \dots + n(n+5) = \frac{1}{3}n(n+1)(n+8)$  can be rewritten as  $\sum_{r=1}^n r(r+5) = \frac{1}{3}n(n+1)(n+8)$ .

$$n = 1; \text{ LHS} = \sum_{r=1}^1 r(r+5) = 1(6) = 6$$

$$\text{RHS} = \frac{1}{3}(1)(2)(9) = \frac{1}{3}(18) = 6$$

As LHS = RHS, the summation formula is true for  $n = 1$ .

Assume that the summation formula is true for  $n = k$ .

$$\text{i.e. } \sum_{r=1}^k r(r+5) = \frac{1}{3}k(k+1)(k+8).$$

With  $n = k + 1$  terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{k+1} r(r+5) &= 1(6) + 2(7) + 3(8) + \dots + k(k+5) + (k+1)(k+6) \\ &= \frac{1}{3}k(k+1)(k+8) + (k+1)(k+6) \\ &= \frac{1}{3}(k+1)[k(k+8) + 3(k+6)] \\ &= \frac{1}{3}(k+1)[k^2 + 8k + 3k + 18] \\ &= \frac{1}{3}(k+1)[k^2 + 11k + 18] \\ &= \frac{1}{3}(k+1)(k+9)(k+2) \\ &= \frac{1}{3}(k+1)(k+2)(k+9) \\ &= \frac{1}{3}(k+1)(k+1+1)(k+1+8) \end{aligned}$$

Therefore, summation formula is true when  $n = k + 1$ .

If the summation formula is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the result is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

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## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise A, Question 5

#### Question:

Prove by the method of mathematical induction, the following statement for  $n \in \mathbb{Z}^+$ .

$$\sum_{r=1}^n r(3r-1) = n^2(n+1)$$

#### Solution:

$$n = 1; \text{LHS} = \sum_{r=1}^1 r(3r-1) = 1(2) = 2$$

$$\text{RHS} = 1^2(2) = (1)(2) = 2$$

As LHS = RHS, the summation formula is true for  $n = 1$ .

Assume that the summation formula is true for  $n = k$ .

$$\text{i.e. } \sum_{r=1}^k r(3r-1) = k^2(k+1).$$

With  $n = k + 1$  terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{k+1} r(3r-1) &= 1(2) + 2(5) + 3(8) + \dots + k(3k-1) + (k+1)(3(k+1)-1) \\ &= k^2(k+1) + (k+1)(3k+3-1) \\ &= k^2(k+1) + (k+1)(3k+2) \\ &= (k+1)[k^2+3k+2] \\ &= (k+1)(k+2)(k+1) \\ &= (k+1)^2(k+2) \\ &= (k+1)^2(k+1+1) \end{aligned}$$

Therefore, summation formula is true when  $n = k + 1$ .

If the summation formula is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the result is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

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## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise A, Question 6

#### Question:

Prove by the method of mathematical induction, the following statement for  $n \in \mathbb{Z}^+$ .

$$\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}n(4n^2-1)$$

#### Solution:

$$\begin{aligned} n = 1; \text{LHS} &= \sum_{r=1}^1 (2r-1)^2 = 1^2 = 1 \\ \text{RHS} &= \frac{1}{3}(1)(4-1) = \frac{1}{3}(1)(3) = 1 \end{aligned}$$

As LHS = RHS, the summation formula is true for  $n = 1$ .

Assume that the summation formula is true for  $n = k$ .

$$\text{ie. } \sum_{r=1}^k (2r-1)^2 = \frac{1}{3}k(4k^2-1) = \frac{1}{3}k(2k+1)(2k-1).$$

With  $n = k + 1$  terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{k+1} (2r-1)^2 &= 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 + (2(k+1)-1)^2 \\ &= \frac{1}{3}k(4k^2-1) + (2k+2-1)^2 \\ &= \frac{1}{3}k(4k^2-1) + (2k+1)^2 \\ &= \frac{1}{3}k(2k+1)(2k-1) + (2k+1)^2 \\ &= \frac{1}{3}(2k+1)[k(2k-1) + 3(2k+1)] \\ &= \frac{1}{3}(2k+1)[2k^2 - k + 6k + 3] \\ &= \frac{1}{3}(2k+1)[2k^2 + 5k + 3] \\ &= \frac{1}{3}(2k+1)(k+1)(2k+3) \\ &= \frac{1}{3}(k+1)(2k+3)(2k+1) \\ &= \frac{1}{3}(k+1)[2(k+1)+1][2(k+1)-1] \\ &= \frac{1}{3}(k+1)[4(k+1)^2 - 1] \end{aligned}$$

Therefore, summation formula is true when  $n = k + 1$ .

If the summation formula is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the result is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.



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## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise A, Question 7

#### Question:

Prove by the method of mathematical induction, the following statement for  $n \in \mathbb{Z}^+$ .

$$\sum_{r=1}^n 2^r = 2^{n+1} - 2$$

#### Solution:

$$\begin{aligned} n = 1; \text{LHS} &= \sum_{r=1}^1 2^r = 2^1 = 2 \\ \text{RHS} &= 2^2 - 2 = 4 - 2 = 2 \end{aligned}$$

As LHS = RHS, the summation formula is true for  $n = 1$ .

Assume that the summation formula is true for  $n = k$ .

$$\text{ie. } \sum_{r=1}^k 2^r = 2^{k+1} - 2.$$

With  $n = k + 1$  terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{k+1} 2^r &= 2^1 + 2^2 + 2^3 + \dots + 2^k + 2^{k+1} \\ &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2(2^{k+1}) - 2 \\ &= 2^1(2^{k+1}) - 2 \\ &= 2^{1+k+1} - 2 \\ &= 2^{k+1+1} - 2 \end{aligned}$$

Therefore, summation formula is true when  $n = k + 1$ .

If the summation formula is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the result is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.



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## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise A, Question 8

#### Question:

Prove by the method of mathematical induction, the following statement for  $n \in \mathbb{Z}^+$ .

$$\sum_{r=1}^n 4^{r-1} = \frac{4^n - 1}{3}$$

#### Solution:

$$\begin{aligned} n = 1; \text{LHS} &= \sum_{r=1}^1 4^{r-1} = 4^0 = 1 \\ \text{RHS} &= \frac{4 - 1}{3} = \frac{3}{3} = 1 \end{aligned}$$

As LHS = RHS, the summation formula is true for  $n = 1$ .

Assume that the summation formula is true for  $n = k$ .

$$\text{ie. } \sum_{r=1}^k 4^{r-1} = \frac{4^k - 1}{3}.$$

With  $n = k + 1$  terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{k+1} 4^{r-1} &= 4^0 + 4^1 + 4^2 + \dots + 4^{k-1} + 4^{k+1-1} \\ &= \frac{4^k - 1}{3} + 4^k \\ &= \frac{4^k - 1}{3} + \frac{3(4^k)}{3} \\ &= \frac{4^k - 1 + 3(4^k)}{3} \\ &= \frac{4(4^k) - 1}{3} \\ &= \frac{4^1(4^k) - 1}{3} \\ &= \frac{4^{k+1} - 1}{3} \end{aligned}$$

Therefore, summation formula is true when  $n = k + 1$ .

If the summation formula is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the result is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

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## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise A, Question 9

#### Question:

Prove by the method of mathematical induction, the following statement for  $n \in \mathbb{Z}^+$ .

$$\sum_{r=1}^n r(r!) = (n+1)! - 1$$

#### Solution:

$$n = 1; \text{LHS} = \sum_{r=1}^1 r(r!) = 1(1!) = 1(1) = 1$$

$$\text{RHS} = 2! - 1 = 2 - 1 = 1$$

As LHS = RHS, the summation formula is true for  $n = 1$ .

Assume that the summation formula is true for  $n = k$ .

$$\text{ie. } \sum_{r=1}^k r(r!) = (k+1)! - 1.$$

With  $n = k + 1$  terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{k+1} r(r!) &= 1(1!) + 2(2!) + 3(3!) + \dots + k(k!) + (k+1)[(k+1)!] \\ &= (k+1)! - 1 + (k+1)[(k+1)!] \\ &= (k+1)! + (k+1)[(k+1)!] - 1 \\ &= (k+1)! [1 + k + 1] - 1 \\ &= (k+1)! (k+2) - 1 \\ &= (k+2)! - 1 \\ &= (k+1+1)! - 1 \end{aligned}$$

Therefore, summation formula is true when  $n = k + 1$ .

If the summation formula is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the result is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

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## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise A, Question 10

#### Question:

Prove by the method of mathematical induction, the following statement for  $n \in \mathbb{Z}^+$ .

$$\sum_{r=1}^{2n} r^2 = \frac{1}{3}n(2n+1)(4n+1)$$

#### Solution:

$$\begin{aligned} n = 1; \text{LHS} &= \sum_{r=1}^2 r^2 = 1^2 + 2^2 = 1 + 4 = 5 \\ \text{RHS} &= \frac{1}{3}(1)(3)(5) = \frac{1}{3}(15) = 5 \end{aligned}$$

As LHS = RHS, the summation formula is true for  $n = 1$ .

Assume that the summation formula is true for  $n = k$ .

$$\text{ie. } \sum_{r=1}^{2k} r^2 = \frac{1}{3}k(2k+1)(4k+1).$$

With  $n = k + 1$  terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{2(k+1)} r^2 &= \sum_{r=1}^{2k+2} r^2 = 1^2 + 2^2 + 3^2 + \dots + k^2 + (2k+1)^2 + (2k+2)^2 \\ &= \frac{1}{3}k(2k+1)(4k+1) + (2k+1)^2 + (2k+2)^2 \\ &= \frac{1}{3}k(2k+1)(4k+1) + (2k+1)^2 + 4(k+1)^2 \\ &= \frac{1}{3}(2k+1)[k(4k+1) + 3(2k+1)] + 4(k+1)^2 \\ &= \frac{1}{3}(2k+1)[4k^2 + 7k + 3] + 4(k+1)^2 \\ &= \frac{1}{3}(2k+1)(4k+3)(k+1) + 4(k+1)^2 \\ &= \frac{1}{3}(k+1)[(2k+1)(4k+3) + 12(k+1)] \\ &= \frac{1}{3}(k+1)[8k^2 + 6k + 4k + 3 + 12k + 12] \\ &= \frac{1}{3}(k+1)[8k^2 + 22k + 15] \\ &= \frac{1}{3}(k+1)(2k+3)(4k+5) \\ &= \frac{1}{3}(k+1)[2(k+1)+1][4(k+1)+1] \end{aligned}$$

Therefore, summation formula is true when  $n = k + 1$ .

If the summation formula is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the result is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.



# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise B, Question 1

#### Question:

Use the method of mathematical induction to prove the following statement for  $n \in \mathbb{Z}^+$ .

$8^n - 1$  is divisible by 7

#### Solution:

Let  $f(n) = 8^n - 1$ , where  $n \in \mathbb{Z}^+$ .

$\therefore f(1) = 8^1 - 1 = 7$ , which is divisible by 7.

$\therefore f(n)$  is divisible by 7 when  $n = 1$ .

Assume that for  $n = k$ ,

$f(k) = 8^k - 1$  is divisible by 7 for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned}\therefore f(k+1) &= 8^{k+1} - 1 \\ &= 8^k \cdot 8^1 - 1 \\ &= 8(8^k) - 1\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [8(8^k) - 1] - [8^k - 1] \\ &= 8(8^k) - 1 - 8^k + 1 \\ &= 7(8^k)\end{aligned}$$

$$\therefore f(k+1) = f(k) + 7(8^k)$$

As both  $f(k)$  and  $7(8^k)$  are divisible by 7 then the sum of these two terms must also be divisible by 7. Therefore  $f(n)$  is divisible by 7 when  $n = k + 1$ .

If  $f(n)$  is divisible by 7 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 7 when  $n = k + 1$ . As  $f(n)$  is divisible by 7 when  $n = 1$ ,  $f(n)$  is also divisible by 7 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise B, Question 2

#### Question:

Use the method of mathematical induction to prove the following statement for  $n \in \mathbb{Z}^+$ .

$3^{2n} - 1$  is divisible by 8

#### Solution:

Let  $f(n) = 3^{2n} - 1$ , where  $n \in \mathbb{Z}^+$ .

$\therefore f(1) = 3^{2(1)} - 1 = 9 - 1 = 8$ , which is divisible by 8.

$\therefore f(n)$  is divisible by 8 when  $n = 1$ .

Assume that for  $n = k$ ,

$f(k) = 3^{2k} - 1$  is divisible by 8 for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned}\therefore f(k+1) &= 3^{2(k+1)} - 1 \\ &= 3^{2k+2} - 1 \\ &= 3^{2k} \cdot 3^2 - 1 \\ &= 9(3^{2k}) - 1\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [9(3^{2k}) - 1] - [3^{2k} - 1] \\ &= 9(3^{2k}) - 1 - 3^{2k} + 1 \\ &= 8(3^{2k})\end{aligned}$$

$$\therefore f(k+1) = f(k) + 8(3^{2k})$$

As both  $f(k)$  and  $8(3^{2k})$  are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore  $f(n)$  is divisible by 8 when  $n = k + 1$ .

If  $f(n)$  is divisible by 8 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 8 when  $n = k + 1$ . As  $f(n)$  is divisible by 8 when  $n = 1$ ,  $f(n)$  is also divisible by 8 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise B, Question 3

#### Question:

Use the method of mathematical induction to prove the following statement for  $n \in \mathbb{Z}^+$ .

$5^n + 9^n + 2$  is divisible by 4

#### Solution:

Let  $f(n) = 5^n + 9^n + 2$ , where  $n \in \mathbb{Z}^+$ .

$\therefore f(1) = 5^1 + 9^1 + 2 = 5 + 9 + 2 = 16$ , which is divisible by 4.

$\therefore f(n)$  is divisible by 4 when  $n = 1$ .

Assume that for  $n = k$ ,

$f(k) = 5^k + 9^k + 2$  is divisible by 4 for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned}\therefore f(k+1) &= 5^{k+1} + 9^{k+1} + 2 \\ &= 5^k \cdot 5^1 + 9^k \cdot 9^1 + 2 \\ &= 5(5^k) + 9(9^k) + 2\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [5(5^k) + 9(9^k) + 2] - [5^k + 9^k + 2] \\ &= 5(5^k) + 9(9^k) + 2 - 5^k - 9^k - 2 \\ &= 4(5^k) + 8(9^k) \\ &= 4[5^k + 2(9)^k]\end{aligned}$$

$$\therefore f(k+1) = f(k) + 4[5^k + 2(9)^k]$$

As both  $f(k)$  and  $4[5^k + 2(9)^k]$  are divisible by 4 then the sum of these two terms must also be divisible by 4. Therefore  $f(n)$  is divisible by 4 when  $n = k + 1$ .

If  $f(n)$  is divisible by 4 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 4 when  $n = k + 1$ . As  $f(n)$  is divisible by 4 when  $n = 1$ ,  $f(n)$  is also divisible by 4 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise B, Question 4

#### Question:

Use the method of mathematical induction to prove the following statement for  $n \in \mathbb{Z}^+$ .

$2^{4n} - 1$  is divisible by 15

#### Solution:

Let  $f(n) = 2^{4n} - 1$ , where  $n \in \mathbb{Z}^+$ .

$\therefore f(1) = 2^{4(1)} - 1 = 16 - 1 = 15$ , which is divisible by 15.

$\therefore f(n)$  is divisible by 15 when  $n = 1$ .

Assume that for  $n = k$ ,

$f(k) = 2^{4k} - 1$  is divisible by 15 for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned}\therefore f(k+1) &= 2^{4(k+1)} - 1 \\ &= 2^{4k+4} - 1 \\ &= 2^{4k} \cdot 2^4 - 1 \\ &= 16(2^{4k}) - 1\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [16(2^{4k}) - 1] - [2^{4k} - 1] \\ &= 16(2^{4k}) - 1 - 2^{4k} + 1 \\ &= 15(8^k)\end{aligned}$$

$$\therefore f(k+1) = f(k) + 15(8^k)$$

As both  $f(k)$  and  $15(8^k)$  are divisible by 15 then the sum of these two terms must also be divisible by 15. Therefore  $f(n)$  is divisible by 15 when  $n = k + 1$ .

If  $f(n)$  is divisible by 15 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 15 when  $n = k + 1$ . As  $f(n)$  is divisible by 15 when  $n = 1$ ,  $f(n)$  is also divisible by 15 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.



# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise B, Question 5

#### Question:

Use the method of mathematical induction to prove the following statement for  $n \in \mathbb{Z}^+$ .

$3^{2n-1} + 1$  is divisible by 4

#### Solution:

Let  $f(n) = 3^{2n-1} + 1$ , where  $n \in \mathbb{Z}^+$ .

$\therefore f(1) = 3^{2(1)-1} + 1 = 3 + 1 = 4$ , which is divisible by 4.

$\therefore f(n)$  is divisible by 4 when  $n = 1$ .

Assume that for  $n = k$ ,

$f(k) = 3^{2k-1} + 1$  is divisible by 4 for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned} \therefore f(k+1) &= 3^{2(k+1)-1} + 1 \\ &= 3^{2k+2-1} + 1 \\ &= 3^{2k-1} \cdot 3^2 + 1 \\ &= 9(3^{2k-1}) + 1 \end{aligned}$$

$$\begin{aligned} \therefore f(k+1) - f(k) &= [9(3^{2k-1}) + 1] - [3^{2k-1} + 1] \\ &= 9(3^{2k-1}) + 1 - 3^{2k-1} - 1 \\ &= 8(3^{2k-1}) \end{aligned}$$

$$\therefore f(k+1) = f(k) + 8(3^{2k-1})$$

As both  $f(k)$  and  $8(3^{2k-1})$  are divisible by 4 then the sum of these two terms must also be divisible by 4. Therefore  $f(n)$  is divisible by 4 when  $n = k + 1$ .

If  $f(n)$  is divisible by 4 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 4 when  $n = k + 1$ . As  $f(n)$  is divisible by 4 when  $n = 1$ ,  $f(n)$  is also divisible by 4 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise B, Question 6

#### Question:

Use the method of mathematical induction to prove the following statement for  $n \in \mathbb{Z}^+$ .

$n^3 + 6n^2 + 8n$  is divisible by 3

#### Solution:

Let  $f(n) = n^3 + 6n^2 + 8n$ , where  $n \geq 1$  and  $n \in \mathbb{Z}^+$ .

$\therefore f(1) = 1 + 6 + 8 = 15$ , which is divisible by 3.

$\therefore f(n)$  is divisible by 3 when  $n = 1$ .

Assume that for  $n = k$ ,

$f(k) = k^3 + 6k^2 + 8k$  is divisible by 3 for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned}\therefore f(k+1) &= (k+1)^3 + 6(k+1)^2 + 8(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 6(k^2 + 2k + 1) + 8(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 6k^2 + 12k + 6 + 8k + 8 \\ &= k^3 + 9k^2 + 23k + 15\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [k^3 + 9k^2 + 23k + 15] - [k^3 + 6k^2 + 8k] \\ &= 3k^2 + 15k + 15 \\ &= 3(k^2 + 5k + 5)\end{aligned}$$

$$\therefore f(k+1) = f(k) + 3(k^2 + 5k + 5)$$

As both  $f(k)$  and  $3(k^2 + 5k + 5)$  are divisible by 3 then the sum of these two terms must also be divisible by 3.

Therefore  $f(n)$  is divisible by 3 when  $n = k + 1$ .

If  $f(n)$  is divisible by 3 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 3 when  $n = k + 1$ . As  $f(n)$  is divisible by 3 when  $n = 1$ ,  $f(n)$  is also divisible by 3 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise B, Question 7

#### Question:

Use the method of mathematical induction to prove the following statement for  $n \in \mathbb{Z}^+$ .

$n^3 + 5n$  is divisible by 6

#### Solution:

Let  $f(n) = n^3 + 5n$ , where  $n \geq 1$  and  $n \in \mathbb{Z}^+$ .

$\therefore f(1) = 1 + 5 = 6$ , which is divisible by 6.

$\therefore f(n)$  is divisible by 6 when  $n = 1$ .

Assume that for  $n = k$ ,

$f(k) = k^3 + 5k$  is divisible by 6 for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned} \therefore f(k+1) &= (k+1)^3 + 5(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 5(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 5k + 5 \\ &= k^3 + 3k^2 + 8k + 6 \end{aligned}$$

$$\begin{aligned} \therefore f(k+1) - f(k) &= [k^3 + 3k^2 + 8k + 6] - [k^3 + 5k] \\ &= 3k^2 + 3k + 6 \\ &= 3k(k+1) + 6 \\ &= 3(2m) + 6 \\ &= 6m + 6 \\ &= 6(m+1) \end{aligned}$$

Let  $k(k+1) = 2m$ ,  $m \in \mathbb{Z}^+$ , as the product of two consecutive integers must be even.

$$\therefore f(k+1) = f(k) + 6(m+1).$$

As both  $f(k)$  and  $6(m+1)$  are divisible by 6 then the sum of these two terms must also be divisible by 6. Therefore  $f(n)$  is divisible by 6 when  $n = k+1$ .

If  $f(n)$  is divisible by 6 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 6 when  $n = k+1$ . As  $f(n)$  is divisible by 6 when  $n = 1$ ,  $f(n)$  is also divisible by 6 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise B, Question 8

#### Question:

Use the method of mathematical induction to prove the following statement for  $n \in \mathbb{Z}^+$ .

$2^n \cdot 3^{2n} - 1$  is divisible by 17

#### Solution:

Let  $f(n) = 2^n \cdot 3^{2n} - 1$ , where  $n \in \mathbb{Z}^+$ .

$\therefore f(1) = 2^1 \cdot 3^{2(1)} - 1 = 2(9) - 1 = 18 - 1 = 17$ , which is divisible by 17.

$\therefore f(n)$  is divisible by 17 when  $n = 1$ .

Assume that for  $n = k$ ,

$f(k) = 2^k \cdot 3^{2k} - 1$  is divisible by 17 for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned} \therefore f(k+1) &= 2^{k+1} \cdot 3^{2(k+1)} - 1 \\ &= 2^k(2)^1(3)^{2k}(3)^2 - 1 \\ &= 2^k(2)^1(3)^{2k}(9) - 1 \\ &= 18(2^k \cdot 3^{2k}) - 1 \end{aligned}$$

$$\begin{aligned} \therefore f(k+1) - f(k) &= [18(2^k \cdot 3^{2k}) - 1] - [2^k \cdot 3^{2k} - 1] \\ &= 18(2^k \cdot 3^{2k}) - 1 - 2^k \cdot 3^{2k} + 1 \\ &= 17(2^k \cdot 3^{2k}) \end{aligned}$$

$$\therefore f(k+1) = f(k) + 17(2^k \cdot 3^{2k})$$

As both  $f(k)$  and  $17(2^k \cdot 3^{2k})$  are divisible by 17 then the sum of these two terms must also be divisible by 17.

Therefore  $f(n)$  is divisible by 17 when  $n = k + 1$ .

If  $f(n)$  is divisible by 17 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 17 when  $n = k + 1$ . As  $f(n)$  is divisible by 17 when  $n = 1$ ,  $f(n)$  is also divisible by 17 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise B, Question 9

#### Question:

$$f(n) = 13^n - 6^n, n \in \mathbb{Z}^+.$$

**a** Express for  $k \in \mathbb{Z}^+$ ,  $f(k+1) - 6f(k)$  in terms of  $k$ , simplifying your answer.

**b** Use the method of mathematical induction to prove that  $f(n)$  is divisible by 7 for all  $n \in \mathbb{Z}^+$ .

#### Solution:

**a**

$$\begin{aligned} f(k+1) &= 13^{k+1} - 6^{k+1} \\ &= 13^k \cdot 13^1 - 6^k \cdot 6^1 \\ &= 13(13^k) - 6(6^k) \end{aligned}$$

$$\begin{aligned} \therefore f(k+1) - 6f(k) &= [13(13^k) - 6(6^k)] - 6[13^k - 6^k] \\ &= 13(13^k) - 6(6^k) - 6(13^k) + 6(6^k) \\ &= 7(13^k) \end{aligned}$$

**b**  $f(n) = 13^n - 6^n$ , where  $n \in \mathbb{Z}^+$ .

$$\therefore f(1) = 13^1 - 6^1 = 7, \text{ which is divisible by } 7.$$

$\therefore f(n)$  is divisible by 7 when  $n = 1$ .

Assume that for  $n = k$ ,

$$f(k) = 13^k - 6^k \text{ is divisible by } 7 \text{ for } k \in \mathbb{Z}^+.$$

$$\text{From (a), } f(k+1) = 6f(k) + 7(13^k)$$

As both  $6f(k)$  and  $7(13^k)$  are divisible by 7 then the sum of these two terms must also be divisible by 7. Therefore  $f(n)$  is divisible by 7 when  $n = k + 1$ .

If  $f(n)$  is divisible by 7 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 7 when  $n = k + 1$ . As  $f(n)$  is divisible by 7 when  $n = 1$ ,  $f(n)$  is also divisible by 7 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise B, Question 10

#### Question:

$$g(n) = 5^{2n} - 6n + 8, n \in \mathbb{Z}^+.$$

**a** Express for  $k \in \mathbb{Z}^+$ ,  $g(k+1) - 25g(k)$  in terms of  $k$ , simplifying your answer.

**b** Use the method of mathematical induction to prove that  $g(n)$  is divisible by 9 for all  $n \in \mathbb{Z}^+$ .

#### Solution:

**a**

$$\begin{aligned} g(k+1) &= 5^{2(k+1)} - 6(k+1) + 8 \\ &= 5^{2k} \cdot 5^2 - 6k - 6 + 8 \\ &= 25(5^{2k}) - 6k + 2 \end{aligned}$$

$$\begin{aligned} \therefore g(k+1) - 25g(k) &= [25(5^{2k}) - 6k + 2] - 25[5^{2k} - 6k + 8] \\ &= 25(5^{2k}) - 6k + 2 - 25(5^{2k}) + 150k - 200 \\ &= 144k - 198 \end{aligned}$$

**b**

$$g(n) = 5^{2n} - 6n + 8, \text{ where } n \in \mathbb{Z}^+.$$

$$\therefore g(1) = 5^2 - 6(1) + 8 = 25 - 6 + 8 = 27, \text{ which is divisible by 9.}$$

$$\therefore g(n) \text{ is divisible by 9 when } n = 1.$$

Assume that for  $n = k$ ,

$$g(k) = 5^{2k} - 6k + 8 \text{ is divisible by 9 for } k \in \mathbb{Z}^+.$$

$$\begin{aligned} \text{From (a), } g(k+1) &= 25g(k) + 144k - 198 \\ &= 25g(k) + 18(8k - 11) \end{aligned}$$

As both  $25g(k)$  and  $18(8k - 11)$  are divisible by 9 then the sum of these two terms must also be divisible by 9. Therefore  $g(n)$  is divisible by 9 when  $n = k + 1$ .

If  $g(n)$  is divisible by 9 when  $n = k$ , then it has been shown that  $g(n)$  is also divisible by 9 when  $n = k + 1$ . As  $g(n)$  is divisible by 9 when  $n = 1$ ,  $g(n)$  is also divisible by 9 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise B, Question 11

#### Question:

Use the method of mathematical induction to prove that  $8^n - 3^n$  is divisible by 5 for all  $n \in \mathbb{Z}^+$ .

#### Solution:

$$f(n) = 8^n - 3^n, \text{ where } n \in \mathbb{Z}^+.$$

$$\therefore f(1) = 8^1 - 3^1 = 5, \text{ which is divisible by 5.}$$

$$\therefore f(n) \text{ is divisible by 5 when } n = 1.$$

Assume that for  $n = k$ ,

$$f(k) = 8^k - 3^k \text{ is divisible by 5 for } k \in \mathbb{Z}^+.$$

$$\begin{aligned} \therefore f(k+1) &= 8^{k+1} - 3^{k+1} \\ &= 8^k \cdot 8^1 - 3^k \cdot 3^1 \\ &= 8(8^k) - 3(3^k) \end{aligned}$$

$$\begin{aligned} \therefore f(k+1) - 3f(k) &= [8(8^k) - 3(3^k)] - 3[8^k - 3^k] \\ &= 8(8^k) - 3(3^k) - 3(8^k) + 3(3^k) \\ &= 5(8^k) \end{aligned}$$

$$\text{From (a), } f(k+1) = f(k) + 5(8^k)$$

As both  $f(k)$  and  $5(8^k)$  are divisible by 5 then the sum of these two terms must also be divisible by 5. Therefore  $f(n)$  is divisible by 5 when  $n = k + 1$ .

If  $f(n)$  is divisible by 5 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 5 when  $n = k + 1$ . As  $f(n)$  is divisible by 5 when  $n = 1$ ,  $f(n)$  is also divisible by 5 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise B, Question 12

#### Question:

Use the method of mathematical induction to prove that  $3^{2n+2} + 8n - 9$  is divisible by 8 for all  $n \in \mathbb{Z}^+$ .

#### Solution:

$$f(n) = 3^{2n+2} + 8n - 9, \text{ where } n \in \mathbb{Z}^+.$$

$$\therefore f(1) = 3^{2(1)+2} + 8(1) - 9$$

$$= 3^4 + 8 - 9 = 81 - 1 = 80, \text{ which is divisible by 8.}$$

$\therefore f(n)$  is divisible by 8 when  $n = 1$ .

Assume that for  $n = k$ ,

$$f(k) = 3^{2k+2} + 8k - 9 \text{ is divisible by 8 for } k \in \mathbb{Z}^+.$$

$$\begin{aligned} f(k+1) &= 3^{2(k+1)+2} + 8(k+1) - 9 \\ &= 3^{2k+2+2} + 8(k+1) - 9 \\ &= 3^{2k+2} \cdot (3^2) + 8k + 8 - 9 \\ &= 9(3^{2k+2}) + 8k - 1 \end{aligned}$$

$$\begin{aligned} \therefore f(k+1) - f(k) &= [9(3^{2k+2}) + 8k - 1] - [3^{2k+2} + 8k - 9] \\ &= 9(3^{2k+2}) + 8k - 1 - 3^{2k+2} - 8k + 9 \\ &= 8(3^{2k+2}) + 8 \\ &= 8[3^{2k+2} + 1] \end{aligned}$$

$$\therefore f(k+1) = f(k) + 8[3^{2k+2} + 1]$$

As both  $f(k)$  and  $8[3^{2k+2} + 1]$  are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore  $f(n)$  is divisible by 8 when  $n = k + 1$ .

If  $f(n)$  is divisible by 8 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 8 when  $n = k + 1$ . As  $f(n)$  is divisible by 8 when  $n = 1$ ,  $f(n)$  is also divisible by 8 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.



# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise B, Question 13

#### Question:

Use the method of mathematical induction to prove that  $2^{6n} + 3^{2n-2}$  is divisible by 5 for all  $n \in \mathbb{Z}^+$ .

#### Solution:

$$f(n) = 2^{6n} + 3^{2n-2}, \text{ where } n \in \mathbb{Z}^+.$$

$$\therefore f(1) = 2^{6(1)} + 3^{2(1)-2} = 2^6 + 3^0 = 64 + 1 = 65, \text{ which is divisible by 5.}$$

$$\therefore f(n) \text{ is divisible by 5 when } n = 1.$$

Assume that for  $n = k$ ,

$$f(k) = 2^{6k} + 3^{2k-2} \text{ is divisible by 5 for } k \in \mathbb{Z}^+.$$

$$\begin{aligned} \therefore f(k+1) &= 2^{6(k+1)} + 3^{2(k+1)-2} \\ &= 2^{6k+6} + 3^{2k+2-2} \\ &= 2^6(2^{6k}) + 3^2(3^{2k-2}) \\ &= 64(2^{6k}) + 9(3^{2k-2}) \end{aligned}$$

$$\begin{aligned} \therefore f(k+1) - f(k) &= [64(2^{6k}) + 9(3^{2k-2})] - [2^{6k} + 3^{2k-2}] \\ &= 64(2^{6k}) + 9(3^{2k-2}) - 2^{6k} - 3^{2k-2} \\ &= 63(2^{6k}) + 8(3^{2k-2}) \\ &= 63(2^{6k}) + 63(3^{2k-2}) - 55(3^{2k-2}) \\ &= 63[2^{6k} + 3^{2k-2}] - 55(3^{2k-2}) \end{aligned}$$

$$\begin{aligned} \therefore f(k+1) &= f(k) + 63[2^{6k} + 3^{2k-2}] - 55(3^{2k-2}) \\ &= f(k) + 63f(k) - 55(3^{2k-2}) \\ &= 64f(k) - 55(3^{2k-2}) \end{aligned}$$

$$\therefore f(k+1) = 64f(k) - 55(3^{2k-2})$$

As both  $64f(k)$  and  $-55(3^{2k-2})$  are divisible by 5 then the sum of these two terms must also be divisible by 5. Therefore  $f(n)$  is divisible by 5 when  $n = k + 1$ .

If  $f(n)$  is divisible by 5 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 5 when  $n = k + 1$ . As  $f(n)$  is divisible by 5 when  $n = 1$ ,  $f(n)$  is also divisible by 5 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise C, Question 1

#### Question:

Given that  $u_{n+1} = 5u_n + 4$ ,  $u_1 = 4$ , prove by induction that  $u_n = 5^n - 1$ .

#### Solution:

$n = 1$ ;  $u_1 = 5^1 - 1 = 4$ , as given.

$n = 2$ ;  $u_2 = 5^2 - 1 = 24$ , from the general statement.

and  $u_2 = 5u_1 + 4 = 5(4) + 4 = 24$ , from the recurrence relation.

So  $u_n$  is true when  $n = 1$  and also true when  $n = 2$ .

Assume that for  $n = k$  that,  $u_k = 5^k - 1$  is true for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned}\text{Then } u_{k+1} &= 5u_k + 4 \\ &= 5(5^k - 1) + 4 \\ &= 5^{k+1} - 5 + 4 \\ &= 5^{k+1} - 1\end{aligned}$$

Therefore, the general statement,  $u_n = 5^n - 1$  is true when  $n = k + 1$ .

If  $u_n$  is true when  $n = k$ , then it has been shown that  $u_n = 5^n - 1$  is also true when  $n = k + 1$ . As  $u_n$  is true for  $n = 1$  and  $n = 2$ , then  $u_n$  is true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise C, Question 2

#### Question:

Given that  $u_{n+1} = 2u_n + 5$ ,  $u_1 = 3$ , prove by induction that  $u_n = 2^{n+2} - 5$ .

#### Solution:

$n = 1$ ;  $u_1 = 2^{1+2} - 5 = 8 - 5 = 3$ , as given.

$n = 2$ ;  $u_2 = 2^4 - 5 = 16 - 5 = 11$ , from the general statement.

and  $u_2 = 2u_1 + 5 = 2(3) + 5 = 11$ , from the recurrence relation.

So  $u_n$  is true when  $n = 1$  and also true when  $n = 2$ .

Assume that for  $n = k$  that,  $u_k = 2^{k+2} - 5$  is true for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned}\text{Then } u_{k+1} &= 2u_k + 5 \\ &= 2(2^{k+2} - 5) + 5 \\ &= 2^{k+3} - 10 + 5 \\ &= 2^{k+1+2} - 5\end{aligned}$$

Therefore, the general statement,  $u_n = 2^{n+2} - 5$  is true when  $n = k + 1$ .

If  $u_n$  is true when  $n = k$ , then it has been shown that  $u_n = 2^{n+2} - 5$  is also true when  $n = k + 1$ . As  $u_n$  is true for  $n = 1$  and  $n = 2$ , then  $u_n$  is true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise C, Question 3

#### Question:

Given that  $u_{n+1} = 5u_n - 8$ ,  $u_1 = 3$ , prove by induction that  $u_n = 5^{n-1} + 2$ .

#### Solution:

$n = 1$ ;  $u_1 = 5^{1-1} + 2 = 1 + 2 = 3$ , as given.

$n = 2$ ;  $u_2 = 5^{2-1} + 2 = 5 + 2 = 7$ , from the general statement.

and  $u_2 = 5u_1 - 8 = 5(3) - 8 = 7$ , from the recurrence relation.

So  $u_n$  is true when  $n = 1$  and also true when  $n = 2$ .

Assume that for  $n = k$  that,  $u_k = 5^{k-1} + 2$  is true for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned}\text{Then } u_{k+1} &= 5u_k - 8 \\ &= 5(5^{k-1} + 2) - 8 \\ &= 5^{k-1+1} + 10 - 8 \\ &= 5^k + 2 \\ &= 5^{k+1-1} + 2\end{aligned}$$

Therefore, the general statement,  $u_n = 5^{n-1} + 2$  is true when  $n = k + 1$ .

If  $u_n$  is true when  $n = k$ , then it has been shown that  $u_n = 5^{n-1} + 2$  is also true when  $n = k + 1$ . As  $u_n$  is true for  $n = 1$  and  $n = 2$ , then  $u_n$  is true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise C, Question 4

#### Question:

Given that  $u_{n+1} = 3u_n + 1$ ,  $u_1 = 1$ , prove by induction that  $u_n = \frac{3^n - 1}{2}$ .

#### Solution:

$$n = 1; u_1 = \frac{3^1 - 1}{2} = \frac{2}{2} = 1, \text{ as given.}$$

$$n = 2; u_2 = \frac{3^2 - 1}{2} = \frac{8}{2} = 4, \text{ from the general statement.}$$

and  $u_2 = 3u_1 + 1 = 3(1) + 1 = 4$ , from the recurrence relation.

So  $u_n$  is true when  $n = 1$  and also true when  $n = 2$ .

Assume that for  $n = k$  that,  $u_k = \frac{3^k - 1}{2}$  is true for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned} \text{Then } u_{k+1} &= 3u_k + 1 \\ &= 3\left(\frac{3^k - 1}{2}\right) + 1 \\ &= \left(\frac{3(3^k) - 3}{2}\right) + \frac{2}{2} \\ &= \frac{3^{k+1} - 3 + 2}{2} \\ &= \frac{3^{k+1} - 1}{2} \end{aligned}$$

Therefore, the general statement,  $u_n = \frac{3^n - 1}{2}$  is true when  $n = k + 1$ .

If  $u_n$  is true when  $n = k$ , then it has been shown that  $u_n = \frac{3^n - 1}{2}$  is also true when  $n = k + 1$ . As  $u_n$  is true for  $n = 1$  and  $n = 2$ , then  $u_n$  is true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise C, Question 5

#### Question:

Given that  $u_{n+2} = 5u_{n+1} - 6u_n$ ,  $u_1 = 1$ ,  $u_2 = 5$  prove by induction that  $u_n = 3^n - 2^n$ .

#### Solution:

$$n = 1; u_1 = 3^1 - 2^1 = 3 - 2 = 1, \text{ as given.}$$

$$n = 2; u_2 = 3^2 - 2^2 = 9 - 4 = 5, \text{ as given.}$$

$$n = 3; u_3 = 3^3 - 2^3 = 27 - 8 = 19, \text{ from the general statement.}$$

$$\text{and } u_3 = 5u_2 - 6u_1 = 5(5) - 6(1)$$

$$= 25 - 6 = 19, \text{ from the recurrence relation.}$$

So  $u_n$  is true when  $n = 1$ ,  $n = 2$  and also true when  $n = 3$ .

Assume that for  $n = k$  and  $n = k + 1$ ,

both  $u_k = 3^k - 2^k$  and  $u_{k+1} = 3^{k+1} - 2^{k+1}$  are true for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned} \text{Then } u_{k+2} &= 5u_{k+1} - 6u_k \\ &= 5(3^{k+1} - 2^{k+1}) - 6(3^k - 2^k) \\ &= 5(3^{k+1}) - 5(2^{k+1}) - 6(3^k) + 6(2^k) \\ &= 5(3^{k+1}) - 5(2^{k+1}) - 2(3^1)(3^k) + 3(2^1)(2^k) \\ &= 5(3^{k+1}) - 5(2^{k+1}) - 2(3^{k+1}) + 3(2^{k+1}) \\ &= 3(3^{k+1}) - 2(2^{k+1}) \\ &= (3^1)(3^{k+1}) - (2^1)(2^{k+1}) \\ &= 3^{k+2} - 2^{k+2} \end{aligned}$$

Therefore, the general statement,  $u_n = 3^n - 2^n$  is true when  $n = k + 2$ .

If  $u_n$  is true when  $n = k$  and  $n = k + 1$  then it has been shown that  $u_n = 3^n - 2^n$  is also true when  $n = k + 2$ . As  $u_n$  is true for  $n = 1, n = 2$  and  $n = 3$ , then  $u_n$  is true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise C, Question 6

#### Question:

Given that  $u_{n+2} = 6u_{n+1} - 9u_n$ ,  $u_1 = -1$ ,  $u_2 = 0$ , prove by induction that  $u_n = (n-2)3^{n-1}$ .

#### Solution:

$$n = 1; u_1 = (1-2)3^{1-1} = (-1)(1) = -1, \text{ as given.}$$

$$n = 2; u_2 = (2-2)3^{2-1} = (0)(3) = 0, \text{ as given.}$$

$$n = 3; u_3 = (3-2)3^{3-1} = (1)(9) = 9, \text{ from the general statement.}$$

$$\begin{aligned} \text{and } u_3 &= 6u_2 - 9u_1 = 6(0) - 9(-1) \\ &= 0 - (-9) = 9, \text{ from the recurrence relation.} \end{aligned}$$

So  $u_n$  is true when  $n = 1$ ,  $n = 2$  and also true when  $n = 3$ .

Assume that for  $n = k$  and  $n = k + 1$ ,

$$\text{both } u_k = (k-2)3^{k-1}$$

$$\text{and } u_{k+1} = (k+1-2)3^{k+1-1} = (k-1)3^k \text{ are true for } k \in \mathbb{Z}^+.$$

$$\begin{aligned} \text{Then } u_{k+2} &= 6u_{k+1} - 9u_k \\ &= 6((k-1)3^k) - 9((k-2)3^{k-1}) \\ &= 6(k-1)(3^k) - 3(k-2) \cdot 3(3^{k-1}) \\ &= 6(k-1)(3^k) - 3(k-2)(3^{k-1+1}) \\ &= 6(k-1)(3^k) - 3(k-2)(3^k) \\ &= (3^k)[6(k-1) - 3(k-2)] \\ &= (3^k)[6k - 6 - 3k + 6] \\ &= 3k(3^k) \\ &= k(3^{k+1}) \\ &= (k+2-2)(3^{k+2-1}) \end{aligned}$$

Therefore, the general statement,  $u_n = (n-2)3^{n-1}$  is true when  $n = k + 2$ .

If  $u_n$  is true when  $n = k$  and  $n = k + 1$  then it has been shown that  $u_n = (n-2)3^{n-1}$  is also true when  $n = k + 2$ . As  $u_n$  is true for  $n = 1$ ,  $n = 2$  and  $n = 3$ , then  $u_n$  is true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise C, Question 7

#### Question:

Given that  $u_{n+2} = 7u_{n+1} - 10u_n$ ,  $u_1 = 1$ ,  $u_2 = 8$ , prove by induction that  $u_n = 2(5^{n-1}) - 2^{n-1}$ .

#### Solution:

$$n = 1; u_1 = 2(5^0) - (2^0) = 2 - 1 = 1, \text{ as given.}$$

$$n = 2; u_2 = 2(5^1) - (2^1) = 10 - 2 = 8, \text{ as given.}$$

$$n = 3; u_3 = 2(5^2) - (2^2) = 50 - 4 = 46, \text{ from the general statement.}$$

$$\begin{aligned} \text{and } u_3 &= 7u_2 - 10u_1 = 7(8) - 10(1) \\ &= 56 - 10 = 46, \text{ from the recurrence relation.} \end{aligned}$$

So  $u_n$  is true when  $n = 1, n = 2$  and also true when  $n = 3$ .

Assume that for  $n = k$  and  $n = k + 1$ ,

$$\text{both } u_k = 2(5^{k-1}) - 2^{k-1}$$

$$\text{and } u_{k+1} = 2(5^{k+1-1}) - 2^{k+1-1} = 2(5^k) - 2^k \text{ are true for } k \in \mathbb{Z}^+.$$

$$\begin{aligned} \text{Then } u_{k+2} &= 7u_{k+1} - 10u_k \\ &= 7(2(5^k) - 2^k) - 10(2(5^{k-1}) - 2^{k-1}) \\ &= 14(5^k) - 7(2^k) - 20(5^{k-1}) + 10(2^{k-1}) \\ &= 14(5^k) - 7(2^k) - 4(5^1)(5^{k-1}) + 5(2^1)(2^{k-1}) \\ &= 14(5^k) - 7(2^k) - 4(5^{k-1+1}) + 5(2^{k-1+1}) \\ &= 14(5^k) - 7(2^k) - 4(5^k) + 5(2^k) \\ &= 2(5^1)(5^k) - (2^1)(2^k) \\ &= 2(5^{k+1}) - (2^{k+1}) \\ &= 2(5^{k+2-1}) - (2^{k+2-1}) \end{aligned}$$

Therefore, the general statement,  $u_n = 2(5^{n-1}) - 2^{n-1}$  is true when  $n = k + 2$ .

If  $u_n$  is true when  $n = k$  and  $n = k + 1$  then it has been shown that  $u_n = 2(5^{n-1}) - 2^{n-1}$  is also true when  $n = k + 2$ . As  $u_n$  is true for  $n = 1, n = 2$  and  $n = 3$ , then  $u_n$  is true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.



# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise C, Question 8

#### Question:

Given that  $u_{n+2} = 6u_{n+1} - 9u_n$ ,  $u_1 = 3$ ,  $u_2 = 36$ , prove by induction that  $u_n = (3n - 2)3^n$ .

#### Solution:

$$n = 1; u_1 = (3(1) - 2)(3^1) = (1)(3) = 3, \text{ as given.}$$

$$n = 2; u_2 = (3(2) - 2)(3^2) = (4)(9) = 36, \text{ as given.}$$

$$n = 3; u_3 = (3(3) - 2)(3^3) = (7)(27) = 189, \text{ from the general statement.}$$

$$\begin{aligned} \text{and } u_3 &= 6u_2 - 9u_1 = 6(36) - 9(3) \\ &= 216 - 27 = 189, \text{ from the recurrence relation.} \end{aligned}$$

So  $u_n$  is true when  $n = 1, n = 2$  and also true when  $n = 3$ .

Assume that for  $n = k$  and  $n = k + 1$ ,

$$\text{both } u_k = (3k - 2)(3^k)$$

and  $u_{k+1} = (3(k+1) - 2)(3^{k+1}) = (3k+1)(3^{k+1})$  are true for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned} \text{Then } u_{k+2} &= 6u_{k+1} - 9u_k \\ &= 6((3k+1)(3^{k+1})) - 9((3k-2)(3^k)) \\ &= 6(3k+1)3^1(3^k) - 9(3k-2)(3^k) \\ &= 18(3k+1)(3^k) - 9(3k-2)(3^k) \\ &= 9(3^k)[2(3k+1) - (3k-2)] \\ &= 9(3^k)[6k+2 - 3k+2] \\ &= 9(3^k)[3k+4] \\ &= 3^2(3^k)[3k+4] \\ &= (3k+4)(3^{k+2}) \\ &= (3(k+2) - 2)(3^{k+2}) \end{aligned}$$

Therefore, the general statement,  $u_n = (3n - 2)3^n$  is true when  $n = k + 2$ .

If  $u_n$  is true when  $n = k$  and  $n = k + 1$  then it has been shown that  $u_n = (3n - 2)3^n$  is also true when  $n = k + 2$ . As  $u_n$  is true for  $n = 1, n = 2$  and  $n = 3$ , then  $u_n$  is true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise D, Question 1

#### Question:

Prove by the method of mathematical induction the following statement for  $n \in \mathbb{Z}^+$ .

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$$

#### Solution:

$$\begin{aligned} n = 1; \text{LHS} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ \text{RHS} &= \begin{pmatrix} 1 & 2(1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

As LHS = RHS, the matrix equation is true for  $n = 1$ .

Assume that the matrix equation is true for  $n = k$ .

$$\text{ie. } \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$$

With  $n = k + 1$  the matrix equation becomes

$$\begin{aligned} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+0 & 2+2k \\ 0+0 & 0+1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2(k+1) \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Therefore the matrix equation is true when  $n = k + 1$ .

If the matrix equation is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the matrix equation is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise D, Question 2

#### Question:

Prove by the method of mathematical induction the following statement for  $n \in \mathbb{Z}^+$ .

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -4n \\ n & -2n+1 \end{pmatrix}$$

#### Solution:

$$\begin{aligned} n = 1; \text{LHS} &= \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^1 = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \\ \text{RHS} &= \begin{pmatrix} 2(1)+1 & -4(1) \\ 1 & -2(1)+1 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

As LHS = RHS, the matrix equation is true for  $n = 1$ .

Assume that the matrix equation is true for  $n = k$ .

$$\text{ie. } \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^k = \begin{pmatrix} 2k+1 & -4k \\ k & -2k+1 \end{pmatrix}$$

With  $n = k + 1$  the matrix equation becomes

$$\begin{aligned} \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^k \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2k+1 & -4k \\ k & -2k+1 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 6k+3-4k & -8k-4+4k \\ 3k-2k+1 & -4k+2k-1 \end{pmatrix} \\ &= \begin{pmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{pmatrix} \\ &= \begin{pmatrix} 2(k+1)+1 & -4(k+1) \\ (k+1) & -2(k+1)+1 \end{pmatrix} \end{aligned}$$

Therefore the matrix equation is true when  $n = k + 1$ .

If the matrix equation is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the matrix equation is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise D, Question 3

#### Question:

Prove by the method of mathematical induction the following statement for  $n \in \mathbb{Z}^+$ .

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$$

#### Solution:

$$\begin{aligned} n = 1; \text{LHS} &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^1 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\ \text{RHS} &= \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

As LHS = RHS, the matrix equation is true for  $n = 1$ .

Assume that the matrix equation is true for  $n = k$ .

$$\text{ie. } \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$$

With  $n = k + 1$  the matrix equation becomes

$$\begin{aligned} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^k \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2(2^k) + 0 & 0 + 0 \\ 2(2^k) - 2 + 1 & 0 + 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^1(2^k) & 0 \\ 2^1(2^k) - 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix} \end{aligned}$$

Therefore the matrix equation is true when  $n = k + 1$ .

If the matrix equation is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the matrix equation is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise D, Question 4

#### Question:

Prove by the method of mathematical induction the following statement for  $n \in \mathbb{Z}^+$ .

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix}$$

#### Solution:

$$\begin{aligned} n = 1; \text{LHS} &= \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^1 = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \\ \text{RHS} &= \begin{pmatrix} 4(1)+1 & -8(1) \\ 2(1) & 1-4(1) \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \end{aligned}$$

As LHS = RHS, the matrix equation is true for  $n = 1$ .

Assume that the matrix equation is true for  $n = k$ .

$$\text{ie. } \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k = \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}.$$

With  $n = k + 1$  the matrix equation becomes

$$\begin{aligned} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^{k+1} &= \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 20k+5-16k & -32k-8+24k \\ 10k+2-8k & -16k-3+12k \end{pmatrix} \\ &= \begin{pmatrix} 4k+5 & -8k-8 \\ 2k+2 & -4k-3 \end{pmatrix} \\ &= \begin{pmatrix} 4(k+1)+1 & -8(k+1) \\ 2(k+1) & 1-4(k+1) \end{pmatrix} \end{aligned}$$

Therefore the matrix equation is true when  $n = k + 1$ .

If the matrix equation is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the matrix equation is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise D, Question 5

#### Question:

Prove by the method of mathematical induction the following statement for  $n \in \mathbb{Z}^+$ .

$$\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 5(2^n - 1) \\ 0 & 1 \end{pmatrix}$$

#### Solution:

$$\begin{aligned} n = 1; \text{LHS} &= \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} \\ \text{RHS} &= \begin{pmatrix} 2^1 & 5(2^1 - 1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

As LHS = RHS, the matrix equation is true for  $n = 1$ .

Assume that the matrix equation is true for  $n = k$ .

$$\text{ie. } \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 2^k & 5(2^k - 1) \\ 0 & 1 \end{pmatrix}$$

With  $n = k + 1$  the matrix equation becomes

$$\begin{aligned} \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^k & 5(2^k - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2(2^k) + 0 & 5(2^k) + 5(2^k - 1) \\ 0 + 0 & 0 + 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^1(2^k) & 5(2^k) + 5(2^k) - 5 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 5(2^1)(2^k) - 5 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 5(2^{k+1}) - 5 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 5(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Therefore the matrix equation is true when  $n = k + 1$ .

If the matrix equation is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the matrix equation is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise E, Question 1

#### Question:

Prove by induction that  $9^n - 1$  is divisible by 8 for  $n \in \mathbb{Z}^+$ .

#### Solution:

Let  $f(n) = 9^n - 1$ , where  $n \in \mathbb{Z}^+$ .

$\therefore f(1) = 9^1 - 1 = 8$ , which is divisible by 8.

$\therefore f(n)$  is divisible by 8 when  $n = 1$ .

Assume that for  $n = k$ ,

$f(k) = 9^k - 1$  is divisible by 8 for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned}\therefore f(k+1) &= 9^{k+1} - 1 \\ &= 9^k \cdot 9^1 - 1 \\ &= 9(9^k) - 1\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [9(9^k) - 1] - [9^k - 1] \\ &= 9(9^k) - 1 - 9^k + 1 \\ &= 8(9^k)\end{aligned}$$

$$\therefore f(k+1) = f(k) + 8(9^k)$$

As both  $f(k)$  and  $8(9^k)$  are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore  $f(n)$  is divisible by 8 when  $n = k + 1$ .

If  $f(n)$  is divisible by 8 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 8 when  $n = k + 1$ . As  $f(n)$  is divisible by 8 when  $n = 1$ ,  $f(n)$  is also divisible by 8 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise E, Question 2

#### Question:

The matrix  $\mathbf{B}$  is given by  $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ .

**a** Find  $\mathbf{B}^2$  and  $\mathbf{B}^3$ .

**b** Hence write down a general statement for  $\mathbf{B}^n$ , for  $n \in \mathbb{Z}^+$ .

**c** Prove, by induction that your answer to part **b** is correct.

#### Solution:

**a**

$$\mathbf{B}^2 = \mathbf{B}\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$$

$$\mathbf{B}^3 = \mathbf{B}^2\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+27 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 27 \end{pmatrix}$$

**b** As  $\mathbf{B}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 3^2 \end{pmatrix}$  and  $\mathbf{B}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 3^3 \end{pmatrix}$ , we suggest that  $\mathbf{B}^n = \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix}$ .

**c**

$$n = 1; \text{LHS} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 1 & 0 \\ 0 & 3^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for  $n = 1$ .

Assume that the matrix equation is true for  $n = k$ .

$$\text{ie. } \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix}$$

With  $n = k + 1$  the matrix equation becomes

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+3(3^k) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 3^{k+1} \end{pmatrix} \end{aligned}$$

Therefore the matrix equation is true when  $n = k + 1$ .

If the matrix equation is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the matrix equation is true for  $n = 1$ , it is



now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

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# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise E, Question 3

#### Question:

Prove by induction that for  $n \in \mathbb{Z}^+$ , that  $\sum_{r=1}^n (3r+4) = \frac{1}{2}n(3n+11)$ .

#### Solution:

$$n = 1; \text{LHS} = \sum_{r=1}^1 (3r+4) = 7$$

$$\text{RHS} = \frac{1}{2}(1)(14) = \frac{1}{2}(14) = 7$$

As LHS = RHS, the summation formula is true for  $n = 1$ .

Assume that the summation formula is true for  $n = k$ .

$$\text{ie. } \sum_{r=1}^k (3r+4) = \frac{1}{2}k(3k+11).$$

With  $n = k + 1$  terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{k+1} (3r+4) &= 7 + 10 + 13 + \dots + (3k+4) + (3(k+1)+4) \\ &= \frac{1}{2}k(3k+11) + (3(k+1)+4) \\ &= \frac{1}{2}k(3k+11) + (3k+7) \\ &= \frac{1}{2}[k(3k+11) + 2(3k+7)] \\ &= \frac{1}{2}[3k^2 + 11k + 6k + 14] \\ &= \frac{1}{2}[3k^2 + 17k + 14] \\ &= \frac{1}{2}(k+1)(3k+14) \\ &= \frac{1}{2}(k+1)[3(k+1)+11] \end{aligned}$$

Therefore, summation formula is true when  $n = k + 1$ .

If the summation formula is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the result is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise E, Question 4

#### Question:

A sequence  $u_1, u_2, u_3, u_4, \dots$  is defined by  $u_{n+1} = 5u_n - 3(2^n)$ ,  $u_1 = 7$ .

**a** Find the first four terms of the sequence.

**b** Prove, by induction for  $n \in \mathbb{Z}^+$ , that  $u_n = 5^n + 2^n$ .

#### Solution:

**a**  $u_{n+1} = 5u_n - 3(2^n)$

Given,  $u_1 = 7$ .

$$u_2 = 5u_1 - 3(2^1) = 5(7) - 6 = 35 - 6 = 29$$

$$u_3 = 5u_2 - 3(2^2) = 5(29) - 3(4) = 145 - 12 = 133$$

$$u_4 = 5u_3 - 3(2^3) = 5(133) - 3(8) = 665 - 24 = 641$$

The first four terms of the sequence are 7, 29, 133, 641.

#### **b**

$$n = 1; u_1 = 5^1 + 2^1 = 5 + 2 = 7, \text{ as given.}$$

$$n = 2; u_2 = 5^2 + 2^2 = 25 + 4 = 29, \text{ from the general statement.}$$

From the recurrence relation in part (a),  $u_2 = 29$ .

So  $u_n$  is true when  $n = 1$  and also true when  $n = 2$ .

Assume that for  $n = k$ ,  $u_k = 5^k + 2^k$  is true for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned} \text{Then } u_{k+1} &= 5u_k - 3(2^k) \\ &= 5(5^k + 2^k) - 3(2^k) \\ &= 5(5^k) + 5(2^k) - 3(2^k) \\ &= 5^1(5^k) + 2^1(2^k) \\ &= 5^{k+1} + 2^{k+1} \end{aligned}$$

Therefore, the general statement,  $u_n = 5^n + 2^n$  is true when  $n = k + 1$ .

If  $u_n$  is true when  $n = k$ , then it has been shown that  $u_n = 5^n + 2^n$  is also true when  $n = k + 1$ . As  $u_n$  is true for  $n = 1$  and  $n = 2$ , then  $u_n$  is true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise E, Question 5

#### Question:

The matrix  $\mathbf{A}$  is given by  $\mathbf{A} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$ .

**a** Prove by induction that  $\mathbf{A}^n = \begin{pmatrix} 8n+1 & 16n \\ -4n & 1-8n \end{pmatrix}$  for  $n \in \mathbb{Z}^+$ .

The matrix  $\mathbf{B}$  is given by  $\mathbf{B} = (\mathbf{A}^n)^{-1}$

**b** Hence find  $\mathbf{B}$  in terms of  $n$ .

#### Solution:

**a**

$$\begin{aligned} n = 1; \text{LHS} &= \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^1 = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix} \\ \text{RHS} &= \begin{pmatrix} 8(1)+1 & 16(1) \\ -4(1) & 1-8(1) \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix} \end{aligned}$$

As LHS = RHS, the matrix equation is true for  $n = 1$ .

Assume that the matrix equation is true for  $n = k$ .

$$\text{ie. } \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^k = \begin{pmatrix} 8k+1 & 16k \\ -4k & 1-8k \end{pmatrix}.$$

With  $n = k + 1$  the matrix equation becomes

$$\begin{aligned} \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^{k+1} &= \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^k \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix} \\ &= \begin{pmatrix} 8k+1 & 16k \\ -4k & 1-8k \end{pmatrix} \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix} \\ &= \begin{pmatrix} 72k+9-64k & 128k+16-112k \\ -36k-4+32k & -64k-7+56k \end{pmatrix} \\ &= \begin{pmatrix} 8k+9 & 16k+16 \\ -4k-4 & -8k-7 \end{pmatrix} \\ &= \begin{pmatrix} 8(k+1)+1 & 16(k+1) \\ -4(k+1) & 1-8(k+1) \end{pmatrix} \end{aligned}$$

Therefore the matrix equation is true when  $n = k + 1$ .

If the matrix equation is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the matrix equation is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

**b**

$$\begin{aligned} \det(\mathbf{A}^n) &= (8n+1)(1-8n) - 64n^2 \\ &= 8n - 64n^2 + 1 - 8n + 64n^2 \\ &= 1 \end{aligned}$$

$$\mathbf{B} = (\mathbf{A}^n)^{-1} = \frac{1}{1} \begin{pmatrix} 1-8n & -16n \\ 4n & 8n+1 \end{pmatrix}$$

$$\text{So } \mathbf{B} = \begin{pmatrix} 1-8n & -16n \\ 4n & 8n+1 \end{pmatrix}$$

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# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise E, Question 6

#### Question:

The function  $f$  is defined by  $f(n) = 5^{2n-1} + 1$ , where  $n \in \mathbb{Z}^+$ .

**a** Show that  $f(n+1) - f(n) = \mu(5^{2n-1})$ , where  $\mu$  is an integer to be determined.

**b** Hence prove by induction that  $f(n)$  is divisible by 6.

#### Solution:

**a**

$$\begin{aligned} f(n+1) &= 5^{2(n+1)-1} + 1 \\ &= 5^{2n+2-1} + 1 \\ &= 5^{2n-1} \cdot 5^2 + 1 \\ &= 25(5^{2n-1}) + 1 \end{aligned}$$

$$\begin{aligned} \therefore f(n+1) - f(n) &= [25(5^{2n-1}) + 1] - [5^{2n-1} + 1] \\ &= 25(5^{2n-1}) + 1 - (5^{2n-1}) - 1 \\ &= 24(5^{2n-1}) \end{aligned}$$

Therefore,  $\mu = 24$ .

**b**  $f(n) = 5^{2n-1} + 1$ , where  $n \in \mathbb{Z}^+$ .

$\therefore f(1) = 5^{2(1)-1} + 1 = 5^1 + 1 = 6$ , which is divisible by 6.

$\therefore f(n)$  is divisible by 6 when  $n = 1$ .

Assume that for  $n = k$ ,

$f(k) = 5^{2k-1} + 1$  is divisible by 6 for  $k \in \mathbb{Z}^+$ .

Using (a),  $f(k+1) - f(k) = 24(5^{2k-1})$

$$\therefore f(k+1) = f(k) + 24(5^{2k-1})$$

As both  $f(k)$  and  $24(5^{2k-1})$  are divisible by 6 then the sum of these two terms must also be divisible by 6. Therefore  $f(n)$  is divisible by 6 when  $n = k + 1$ .

If  $f(n)$  is divisible by 6 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 6 when  $n = k + 1$ . As  $f(n)$  is divisible by 6 when  $n = 1$ ,  $f(n)$  is also divisible by 6 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise E, Question 7

#### Question:

Use the method of mathematical induction to prove that  $7^n + 4^n + 1$  is divisible by 6 for all  $n \in \mathbb{Z}^+$ .

#### Solution:

Let  $f(n) = 7^n + 4^n + 1$ , where  $n \in \mathbb{Z}^+$ .

$\therefore f(1) = 7^1 + 4^1 + 1 = 7 + 4 + 1 = 12$ , which is divisible by 6.

$\therefore f(n)$  is divisible by 6 when  $n = 1$ .

Assume that for  $n = k$ ,

$f(k) = 7^k + 4^k + 1$  is divisible by 6 for  $k \in \mathbb{Z}^+$ .

$$\begin{aligned}\therefore f(k+1) &= 7^{k+1} + 4^{k+1} + 1 \\ &= 7^k \cdot 7^1 + 4^k \cdot 4^1 + 1 \\ &= 7(7^k) + 4(4^k) + 1\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [7(7^k) + 4(4^k) + 1] - [7^k + 4^k + 1] \\ &= 7(7^k) + 4(4^k) + 1 - 7^k - 4^k - 1 \\ &= 6(7^k) + 3(4^k) \\ &= 6(7^k) + 3(4^{k-1}) \cdot 4^1 \\ &= 6(7^k) + 12(4^{k-1}) \\ &= 6[7^k + 2(4)^{k-1}]\end{aligned}$$

$$\therefore f(k+1) = f(k) + 6[7^k + 2(4)^{k-1}]$$

As both  $f(k)$  and  $6[7^k + 2(4)^{k-1}]$  are divisible by 6 then the sum of these two terms must also be divisible by 6.

Therefore  $f(n)$  is divisible by 6 when  $n = k + 1$ .

If  $f(n)$  is divisible by 6 when  $n = k$ , then it has been shown that  $f(n)$  is also divisible by 6 when  $n = k + 1$ . As  $f(n)$  is divisible by 6 when  $n = 1$ ,  $f(n)$  is also divisible by 6 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise E, Question 8

#### Question:

A sequence  $u_1, u_2, u_3, u_4, \dots$  is defined by  $u_{n+1} = \frac{3u_n - 1}{4}$ ,  $u_1 = 2$ .

**a** Find the first five terms of the sequence.

**b** Prove, by induction for  $n \in \mathbb{Z}^+$ , that  $u_n = 4\left(\frac{3}{4}\right)^n - 1$ .

#### Solution:

$$\mathbf{a} \quad u_{n+1} = \frac{3u_n - 1}{4}.$$

Given,  $u_1 = 2$

$$u_2 = \frac{3u_1 - 1}{4} = \frac{3(2) - 1}{4} = \frac{5}{4}$$

$$u_3 = \frac{3u_2 - 1}{4} = \frac{3\left(\frac{5}{4}\right) - 1}{4} = \frac{\frac{11}{4}}{4} = \frac{11}{16}$$

$$u_4 = \frac{3u_3 - 1}{4} = \frac{3\left(\frac{11}{16}\right) - 1}{4} = \frac{\frac{17}{16}}{4} = \frac{17}{64}$$

$$u_5 = \frac{3u_4 - 1}{4} = \frac{3\left(\frac{17}{64}\right) - 1}{4} = \frac{-\frac{13}{64}}{4} = -\frac{13}{256}$$

The first five terms of the sequence are  $2, \frac{5}{4}, \frac{11}{16}, \frac{17}{64}, -\frac{13}{256}$ .

#### **b**

$$n = 1; u_1 = 4\left(\frac{3}{4}\right)^1 - 1 = 3 - 1 = 2, \text{ as given.}$$

$$n = 2; u_2 = 4\left(\frac{3}{4}\right)^2 - 1 = \frac{9}{4} - 1 = \frac{5}{4}, \text{ from the general statement.}$$

From the recurrence relation in part (a),  $u_2 = \frac{5}{4}$ .

So  $u_n$  is true when  $n = 1$  and also true when  $n = 2$ .

Assume that for  $n = k$ ,  $u_k = 4\left(\frac{3}{4}\right)^k - 1$  is true for  $k \in \mathbb{Z}^+$ .



$$\begin{aligned}\text{Then } u_{k+1} &= \frac{3u_k - 1}{4} \\ &= \frac{3\left[4\left(\frac{3}{4}\right)^k - 1\right] - 1}{4} \\ &= \frac{3}{4}\left[4\left(\frac{3}{4}\right)^k - 1\right] - \frac{1}{4} \\ &= 4\left(\frac{3}{4}\right)^1\left(\frac{3}{4}\right)^k - \frac{3}{4} - \frac{1}{4} \\ &= 4\left(\frac{3}{4}\right)^{k+1} - 1\end{aligned}$$

Therefore, the general statement,  $u_n = 4\left(\frac{3}{4}\right)^n - 1$  is true when  $n = k + 1$ .

If  $u_n$  is true when  $n = k$ , then it has been shown that  $u_n = 4\left(\frac{3}{4}\right)^n - 1$  is also true when  $n = k + 1$ . As  $u_n$  is true for  $n = 1$  and  $n = 2$ , then  $u_n$  is true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise E, Question 9

#### Question:

A sequence  $u_1, u_2, u_3, u_4, \dots$  is defined by  $u_n = 3^{2n} + 7^{2n-1}$ .

**a** Show that  $u_{n+1} - 9u_n = \lambda(7^{2k-1})$ , where  $\lambda$  is an integer to be determined.

**b** Hence prove by induction that  $u_n$  is divisible by 8 for all positive integers  $n$ .

#### Solution:

**a**

$$\begin{aligned} u_{n+1} &= 3^{2(n+1)} + 7^{2(n+1)-1} \\ &= 3^{2n}(3^2) + 7^{2n+2-1} \\ &= 3^{2n}(3^2) + 7^{2n-1}(7^2) \\ &= 9(3^{2n}) + 49(7^{2n-1}) \end{aligned}$$

$$\begin{aligned} \therefore u_{n+1} - 9u_n &= [9(3^{2n}) + 49(7^{2n-1})] - 9[3^{2n} + 7^{2n-1}] \\ &= 9(3^{2n}) + 49(7^{2n-1}) - 9(3^{2n}) - 9(7^{2n-1}) \\ &= 40(7^{2n-1}) \end{aligned}$$

Therefore,  $\lambda = 40$ .

**b**  $u_n = 3^{2n} + 7^{2n-1}$ , where  $n \in \mathbb{Z}^+$ .

$\therefore u_1 = 3^{2(1)} + 7^{2(1)-1} = 3^2 + 7^1 = 16$ , which is divisible by 8.

$\therefore u_n$  is divisible by 8 when  $n = 1$ .

Assume that for  $n = k$ ,

$u_k = 3^{2k} + 7^{2k-1}$  is divisible by 8 for  $k \in \mathbb{Z}^+$ .

Using (a),  $u_{k+1} - 9u_k = 40(7^{2k-1})$

$$\therefore u_{k+1} = 9u_k + 40(7^{2k-1})$$

As both  $9u_k$  and  $40(7^{2k-1})$  are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore  $u_n$  is divisible by 8 when  $n = k + 1$ .

If  $u_n$  is divisible by 8 when  $n = k$ , then it has been shown that  $u_n$  is also divisible by 8 when  $n = k + 1$ . As  $u_n$  is divisible by 8 when  $n = 1$ ,  $u_n$  is also divisible by 8 for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.

# Solutionbank FP1

## Edexcel AS and A Level Modular Mathematics

### Proof by mathematical induction

#### Exercise E, Question 10

#### Question:

Prove by induction, for all positive integers  $n$ , that

$$(1 \times 5) + (2 \times 6) + (3 \times 7) + \dots + n(n+4) = \frac{1}{6}n(n+1)(2n+13).$$

#### Solution:

The identity  $(1 \times 5) + (2 \times 6) + (3 \times 7) + \dots + n(n+4) = \frac{1}{6}n(n+1)(2n+13)$ .

can be rewritten as  $\sum_{r=1}^n r(r+4) = \frac{1}{6}n(n+1)(2n+13)$ .

$$\begin{aligned} n = 1; \text{LHS} &= \sum_{r=1}^1 r(r+4) = 1(5) = 5 \\ \text{RHS} &= \frac{1}{6}(1)(2)(15) = \frac{1}{6}(30) = 5 \end{aligned}$$

As LHS = RHS, the summation formula is true for  $n = 1$ .

Assume that the summation formula is true for  $n = k$ .

$$\text{ie. } \sum_{r=1}^k r(r+4) = \frac{1}{6}k(k+1)(2k+13).$$

With  $n = k + 1$  terms the summation formula becomes:

$$\begin{aligned} \sum_{r=1}^{k+1} r(r+4) &= 1(5) + 2(6) + 3(7) + \dots + k(k+4) + (k+1)(k+5) \\ &= \frac{1}{6}k(k+1)(2k+13) + (k+1)(k+5) \\ &= \frac{1}{6}(k+1)[k(2k+13) + 6(k+5)] \\ &= \frac{1}{6}(k+1)[2k^2 + 13k + 6k + 30] \\ &= \frac{1}{6}(k+1)[2k^2 + 19k + 30] \\ &= \frac{1}{6}(k+1)(k+2)(2k+15) \\ &= \frac{1}{6}(k+1)(k+1+1)[2(k+1)+13] \end{aligned}$$

Therefore, summation formula is true when  $n = k + 1$ .

If the summation formula is true for  $n = k$ , then it is shown to be true for  $n = k + 1$ . As the result is true for  $n = 1$ , it is now also true for all  $n \geq 1$  and  $n \in \mathbb{Z}^+$  by mathematical induction.