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Research Article

On Characterization of Various Finite Subgroups of Abelian Groups

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Abstract

The present paper, we characterize finite subgroups. Throughout G always denote a finite group. Let H be a subgroup group of G. We have $H \geq H \cap H^x \geq 1$, for any $x \in G$. We call H to be a TI-subgroup of G if $H \cap H^x = H$ or 1 for any $x \in G$. We have shown that if H is normal in G or if H is of a prime order, then H is a TI-subgroup.

Keywords: Group; Finite group; TI-subgroup; Abelian group.

Introduction

Characterizations in group theory regarding subgroups have been done over a period of time by many authors [1-5]. A topic of some interest is to investigate the finite groups in which certain subgroups are assumed to be TI-subgroups. The author in [6] classified the finite groups all of whose subgroups are TI-subgroups. In [7, 8], Guo, they classified the finite groups whose abelian subgroups are TI-subgroups. The aim of this paper is to study the finite AQTI-groups, that is, all of whose abelian subgroups are QTI (that means quasi-trivialintersection)-subgroups [9]. We obtain a classification of the AQTI-groups in Theorem 3.3 (nilpotent case) and Theorem 3.7 (non-nilpotent case). The aim of this work is to characterize TI, ATI and QTI subgroups in depth.

Research Methodology

Definition 1.1

A subgroup H of G is called a QTI-subgroup if $C_G(x) \leq N_G(H)$ for any $1 \neq x \in H$.

Clearly a TI-subgroup is a QTI-subgroup. However, the converse is not true [10].

Example 1.2

Let V be an elementary abelian 3-group of order 35 and H be a subgroup of GL(5, 3) of order 11^2. Let G = HV, where H acts on V in a natural way. Since 11 does not divides $3^a - 1$ for any a < 5, the actions of H and its nonidentity subgroups on V are irreducible and fixed-point-free. It follows that N_G(W) = V for any proper subgroup W of V and that $C_G(w)$ = V for any $1 \neq w \in W$, and therefore W is a QTI-subgroup of G. In fact, it is not difficult to see that all abelian subgroups of G are QTI-subgroups, and therefore G is an AQTI-group. Let W_0 be a subgroup of V of order 3^4. Since $|W_0 \cap W_0^x|$ = 3^3 for any $1 \neq x \in H$, W_0 is not a TI-subgroup.

A very important question to ask at this juncture is**:** Under which additional condition P, a QTI-subgroup is necessary a TI-subgroup? that is, QTI-subgroup+ P?= TI-subgroup.

Results and discussion

Lemma 3.1

Let G be an AQTI-group. Then the following statements hold.

(i) Any subgroup of G is again an AQTI-group.

(ii) For any abelian subgroup H of G, if H $\cap$ Z(G) > 1, then H is normal in G.

(iii) For any $1 \neq x \in G$, $C_G(x)$ is nilpotent.

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Proof **:** (i) and (ii) are clear. (iii) For any cyclic subgroup $A/\langle x\rangle$ of $C_G(x)/\langle x\rangle$, A is an abelian subgroup of an AQTI group C_G(x), and so A is normal in C_G(x) (see (2)). It follows that all cyclic subgroups (and so all subgroups) of $C_G(x)/\langle x\rangle$ are normal in $C_G(x)/\langle x\rangle$. Then $C_G(x)/\langle x\rangle$ is nilpotent, and so C_G(x) is nilpotent. Recall that a CN-group is a group in which the centralizer of any nonidentity element is nilpotent. Now the above lemma implies that an AQTI group is a CN-group. For any finite group G, we define its prime graph Γ(G) (see [8]) as follows: Whose vertex set is π(G), and two vertices p, q are jointed by an edge if G has an element of order pq. If σ is a vertex set of a connected component of Γ(G), then σ is called a prime component of G. This completes the proof.

Lemma 3.2

([2, Theorem 2.2]) Let G be a CN-group and σ a prime component of G. Then G possesses a nilpotent Hall σ-subgroup H, and any σ-subgroup is contained in some G-conjugate of H, furthermore H is a TI-subgroup if in addition $|\sigma|\geq 2$. In particular, if G is a nonnilpotent AQTI-group, then Γ(G) is disconnected.

We note that the original proof of above lemma is elementary. Recall that a Hamiltonian group is a nonabelian group in which all subgroups are normal. It is known that a Hamiltonian group is a direct product of Q_8, an elementary abelian 2-group and an abelian group of odd order. For a *p*-group *G*, we put $V_1(G)=\langle x^p | x\in G\rangle$.

Theorem 3.3

For a finite p-group G, the following statements are equivalent.

(1) All subgroups of G are TI-subgroups.

(2) All abelian subgroups of G are TI-subgroups.

(3) All abelian subgroups of G are QTI-subgroups, ie., G is an AQTI-group.

(4) G is one of the following p-groups:

(4.1) G is an abelian p-group.

(4.2) G is a Hamiltonian 2-group, that is a product of Q_8 and an elementary abelian 2-group.

(4.3) G is the central product of Q_8 and D_8;

(4.4) G/Z(G) is of order p^2, Z(G) is cyclic and $G' \cong Z_p$ is the only minimal normal subgroup of G.

Remark: The objective of the paper [6] is to show the following: The finite *p*-groups all of whose abelian subgroups are TI-subgroups, are just the groups of types (4.1)-(4.4). Our arguments (of Theorem 3.3) are much shorter than those in [6].

Proof : We need only to show (3) implying (4). Suppose that all abelian subgroups of *G* are normal. Then all subgroups of *G* are normal, and so *G* is of type (4.1) or type (4.2). In what follows we assume that *G* has an abelian but not normal subgroup, and we will show that *G* is of type (4.3) or type (4.4). Observe first that for any nontrivial abelian subgroup *A* of *G*, *A* is normal in *G* iff $A\cap Z(G)>1$ (see Lemma 3.1(ii)).

Step 1. *Z(G)* is cyclic. Suppose that *Z(G)* is not cyclic and let *A* be any abelian subgroup of *G*. If $A\cap Z(G)>1$, then *A* is normal in *G*. If $A\cap Z(G)=1$, then *AU*, *AV* are normal in *G* where *U*, $V\cong Z_p$ are distinct subgroups of *Z(G)*, and so *A* = *AU* ∩ *AV* is normal. This implies that all abelian subgroups are normal, which contradicts our assumption.

Step 2. Let Z be the unique minimal normal subgroup of *G*. Then *G/Z* is abelian, and *Z* = G'. Let *A/Z* be any cyclic subgroup of *G/Z*. Then *A* is normal in *G* because *A* is abelian with $A\cap Z(G)\geq Z$. It follows that all subgroups of *G/Z* are normal. Suppose *G/Z* is nonabelian. Then *G* is a Hamiltonian 2-group, and so $G/Z\cong Q_8\times Z_2\times ...\times Z_2$. Let $T/Z\cong Q_8$. Clearly *T* is normal in *G* and so T' is normal in *G*. Since *Z* is the unique minimal normal subgroup of *G*, $T'\geq Z$, and this implies that $|T/T'|=4$. Now applying [3, Ch3, theorem, 11.9], we conclude that *Z(T)* = *Z*. By [3, Page 94, exercise 58], we get a contradiction. Thus *G/Z* is abelian, and so $Z=G'$.

Step 3. Final part of proof. Since G' = *Z* is the unique minimal normal subgroup of *G*, it follows by [5, Lemma 12.3] that *G/Z(G)* is elementary abelian and that all nonlinear irreducible complex characters of G have degree $\sqrt{|G/Z(G)|}$.

Since G has an abelian but not normal subgroup A and $A \cap Z(G) = 1$, we can find an element t such that $\langle t \rangle \cap Z(G) = 1$. Then $H =: C_G(t) < G$. ...... It is easy to see that H is a maximal subgroup of G and that all abelian subgroups of H are normal (and so H is abelian or $H = Q_8 \times Z_2 \times ... \times Z_2$). Suppose that H is abelian. Since $|G : H| = p$, all nonlinear irreducible complex characters of G have degree p, and this implies that $|G/Z(G)| = p^2$, thus G is of type (4.4). Suppose that $H = Q_8 \times Z_2 \times ... \times Z_2$. Then G possesses an abelian subgroup of index 4. It follows that all nonlinear irreducible complex characters of G have degree 2 or 4. Thus either $|G/Z(G)| = 4$ and then G is of type (4.4), or $|G/Z(G)| = 2^4$. Let us investigate the case when $|G/Z(G)| = 2^4$. For this case, ... we can prove that G is an extra special 2-group of order 2^5 (Thus, $G \cong D_8 * D_8$ or $D_8 * D_8$) and that the case $G = D_8 * D_8$ is impossible. And hence G is a central product of D_8 and Q_8, ie., G is of type (4.3).

Lemma 3.4

Let G be a finite group. Then G is an AQTI-subgroup iff G satisfies the following conditions:

(1) G is a CN-group,

(2) Let σ be any prime component of G and let M be a Hall σ- subgroup of G. Then either M is one of the p-groups listed in theorem 3.3, or M is abelian, or M is a Hamiltonian group.

Applying Theorem 3.3 and Lemma 3.4, we obtain the following result.

Theorem 3.5

Let G be a nilpotent group. Then G is an AQTI-group if and only if one of the following holds.

(1) G is abelian.

(2) G is a Hamiltonian group.

(3) G is of type (4.3) or (4.4) in Theorem 3.1.

The proof of Lemma 3.4: Suppose that G is an AQTI-group. By Lemma 2.2, G is a CN-group, and G possesses a nilpotent Hall σ- subgroup M for any prime component σ of G. Clearly M is again an AQTI-subgroup, and we need to show that if $|\sigma| \geq 2$ then all subgroups of M are normal in M. Assume this is not true. Write M = P × Q, where Q is a nontrivial p'-group, and $P \in Syl_p(M)$ has an abelian but not normal subgroup P_1. Let $1 \neq x \in Z(Q) \leq P_1 \times Q$. As $P_1 \times Z(Q)$ is a QTI-subgroup of M, M = $C_M(x) \leq N_M(P_1 \times Z(Q)) = N_P(P_1) \times Q$, and this implies that P_1 is normal in P, a contradiction. Suppose conversely that G satisfies the conditions of Lemma 3.2. Let H be an abelian subgroup of G and $1 \neq x \in H$. Let p be a prime divisor of |H| and let σ be a prime component containing p of G. By Lemma 2.2 we may assume $C_G(x) \leq M$. If $|\sigma| \geq 2$, then M is a Hamiltonian group or an abelian group, thus H is normal in M, and so $C_G(x) = C_M(x) \leq M = N_M(H) \leq N_G(H)$. If $|\sigma| = 1$, then M is an AQTI-group of prime power order, so $C_G(x) = C_M(x) \leq N_M(H) \leq N_G(H)$. Thus H is a QTI-subgroup of G, and therefore G is an AQTI-group. If G = HN is a Frobenius group with a kernel N and a complement H, then we say that H acts frobeniusly on N. In this case, we know that N is nilpotent and any Sylow subgroup of H is either a cyclic group or a generalized quaternion group, and that π(H), π(N) are just two prime components of G (see [8]). If there are M, N < G such that G/N is a Frobenius group with M/N as its kernel and M is a Frobenius group with N as its kernel, then G is called a 2-Frobenius group, and such a 2-Frobenius group is denoted by $Frob_2(G,M,N)$. In this case, we know that G is solvable, and that π(M/N) and π(G/M) $\cup \pi$(N) are just two prime components of G (see [8]).

Lemma 3.6

Let G = HN be a Frobenius group with a complement H and a kernel N. If G is an AQTI-group, then the following statements hold.

(1) H is either a cyclic group or a product of Q_8 with a cyclic group of odd order.

(2) N is either an abelian group or of type (4.4) listed in Theorem 3.3.

Proof: Since G is a Frobenius group, Γ(G) has just two connected components with π(H), π(N) as its vertex sets.

(1) If H is nonnilpotent, then Lemma 2.2 implies that Γ(H) is disconnected, and then Γ(G) has at least three connected components, a contradiction. Thus H is nilpotent. If $P \in$ Syl(H) is not cyclic, then P is a generalized quaternion group, and then $P \cong = Q_8$ by Theorem 3.1. The result follows.

(2) Since N is the Frobenius kernel, N is nilpotent. Assume that N is nonabelian and let P be a nonabelian Sylow p-subgroup of N. Then P is one of the three types listed in Theorem 3.1. Assume that $P \cong Q_8 \times Z_2 \times ... \times Z_2$. Then $V_1(P)$ is a normal subgroup of G of order 2, which is clearly impossible. Assume that P is the central product of Q_8 and D_8. Then Z(P) lies in Z(G), a contradiction. Thus P is of type (4.4) in Theorem 3.3, and then N = P by Theorem 3.3.

Lemma 3.7

Let G = $Frob_2$ (G,H,K). If G is an AQTI-subgroup, then G is isomorphic to symmetric group S_4.

Proof: Note that G is solvable with just two prime components $\pi_1 = \pi(H/K)$ and $\pi_2 = \pi(G) - \pi_1$, and that G has a nilpotent Hall π_2- subgroup W (see Lemma 3.2). Clearly K is the Fitting subgroup of G, thus $C_W(K) \le C_G(K) \le K$, and so W > K > Z(W). Let $p \in \pi$ (G/H) and P be a Sylow p-subgroup of W. Since K >Z(W)$\ge$Z(P), $P \cap K \ge Z(P)$ is nontrivial. Let G_1 > P be a $\pi_1 \cup \{p\}$- Hall subgroup of G. It follows that $G_1 = Frob_2(G_1, H \cap G_1, P \cap K)$. Assume that G_1 < G. Then induction yields that $G_1 \cong S_4$, thus $P \in Syl_2(S_4)$ is isomorphic to D_8, and then W = P by Theorem 3.3, so $G_1 \cong S_4$ as wanted. In what follows, we assume that π_2= {p}. Then W is one of the nonabelian p groups listed in Theorem 3.3.

Case 1. Assume that $W \cong Q_8 \times Z_2 \times ... \times Z_2$. As W > K > Z(W), K is a product of Z_4 and an elementary abelian 2-group. It follows that $V_1(K) < G$ with $|V_1(K)| = 2$, a contradiction.

Case 2. Assume that W is the central product of Q_8 and D_8. As W > K > Z(W), |K| $\in$ {4, 8, 16}. If K is abelian, then K $\in$ {$Z_4 \times Z_2, Z_4, Z_2 \times Z_2$} (see [3, Ch3, Theorem 13.8]). Now $K/\Phi(K) = Z_2$ or $Z_2 \times Z_2$, it follows that $G/K \le Aut(K/\Phi(K)) \le S_3$, then $|P| \le 16$, a contradiction. If K is nonabelian and of order 16, then $K \cong Q_8 \times Z_2$ or $|K/Z(K)| = 4$ with $Z(K) \cong Z_4$. For the first case, let Z = $V_1(K)$; and for the second case, let Z = $V_1(Z(K))$. Then Z is normal in G with |Z| = 2, a contradiction. If K is nonabelian and of order 8, then $K \cong Q_8$ or D_8, and then $G/K \le Aut(K/\Phi(K)) = Aut(Z_2 \times Z_2) = S_3$, thus |P| = 16, a contradiction.

Case 3. Assume that $W/Z(W) \cong Z_p \times Z_p$ and Z(W) is cyclic. Then K is abelian with |W : K| = |K : Z(W)| = p. Note that $G = N_G(U)H = N_G(U)K$ by Frattini argument, where U is a Hall π_1- subgroup of G. Clearly $N_G(U) \cap K = N_K(U) = 1$, and so $N_G(U) \cong G/K$ is a Frobenius group with a complement of order p. Suppose K is not elementary abelian. Then $V_1(K)$ is a nontrivial cyclic normal subgroup of G. Let us consider $G_1 = N_G(U)V_1(K)$. We see that $V_1(K) = Fit(G_1)$, and $N_G(U) \le Aut(V_1(K))$ is abelian, a contradiction. Hence K is elementary abelian, and in particular $Z(W) \cong Z_p$. Now $N_G(U) \le Aut(K) = Aut(Z_p \times Z_p) = GL(2, p)$. Note that if p > 2, then it is easy to check that GL(2, p) has no subgroup which is a Frobenius group with a complement of order p. This implies that $K \cong Z_2 \times Z_2$, and hence $N_G(U) \cong S_3$, and $G \cong S_4$.

Theorem 3.8

Let G be a nonnilpotent group. Then G is an AQTI-subgroup iff G is one of the following groups.

(1) G = HN is a Frobenius group with a complement H and a kernel N, where N is abelian, and H is either a cyclic group or a product of Q_8 with a cyclic group of odd order.

(2) G = HN is a Frobenius group with a complement H and a kernel N, where H is a cyclic subgroup of Z_{p-1} and N is a p-group of the type (4.3) in Theorem 3.3.

(3) $G \cong S_4$.

(4) $G \cong L_2(q)$, where q = 5, 7, 9.

Proof: Suppose that $G \in \{S_4, L_2(5), L_2(7), L_2(9)\}$. Then it is easy to check that G is an AQTI-group. Suppose that G is a Frobenius group of type (1) or (2). We also conclude by Lemma 3.2 that G is an AQTI-group. Suppose that G is a nonnilpotent AQTI-group. Then the prime graph Γ(G) is disconnected (see Lemma 3.2). Assume that G is solvable. It is well known that G is a Frobenius or 2-Frobenius group (see [8]), and then Lemma 3.6 and Lemma 3.7 imply that G is of type (1) or type (2). In what follows, we assume that G is a nonsolvable AQTI-group. Let N = Sol(G), the maximal normal solvable subgroup of G. It follows by [8] that G has a normal series N < H < G such that N and G/H are π-groups and H/N is a nonabelian simple group, where π is the prime component of G containing 2. Furthermore, N = Sol(G) = Fit(G), $G/N \le Aut(H/N)$. Let P_1 be a nilpotent Hall π-subgroup of G (see Lemma 3.2), and $P = P_1 \cap H$.

Claim 1. If N > 1, then π = {2}. Suppose that N > 1 and $|\pi|\geq 2$. By Lemma 3.2, P_1 is a TI-subgroup of G. Since $N\leq P_1$ is a nontrivial normal subgroup of G, P_1 is normal in G, so G is solvable, a contradiction. Thus $|\pi|$ = 1 and so π = {2}.

Claim 2. N = 1. Suppose that N > 1 and let E be any normal subgroup of G with 1 < E $\leq$ N. By claim 1, π = {2} and P is a 2-group. Assume that $C_G(E)N > N$. Since H/N is simple and is a unique minimal normal subgroup of G/N, $C_G(E)N\geq H$. Then any odd order subgroup of H acts trivially on E, which is clearly impossible. Hence $C_G(E)\leq N$, and in particular P > N > Z(P). Now P is one of the 2-groups listed in Theorem 3.1. Arguing as in the proof of Lemma 4.2, we can find a normal subgroup E of G with $1 < E\leq N$ and $E \leq Z_2 \times Z_2$. It follows that $G/C_G(E) \leq$ Aut(E) is solvable, and so G/N is solvable because $C_G(E) \leq N$, a contradiction.

Claim 3. H $\cong$ $L_2(q)$, where q = 5, 7, 9. As N = 1, H is a nonabelian simple group. Since H is an AQTI-group, by Lemma 3.1(iii) H is a CN-group. Note that the only simple nonabelian CN-groups are Sz(q), $L_3(4)$, $L_2(9)$, and $L_2(p)$ where p is a Fermat or a Mersenne prime (see [4, ChXI, Remark 3.12]). Assume that H _= Sz(q). Then |P| = q2, q = 22m+1, where P0 _=_(P) = Z(P) is an elementary abelian group of order q. Checking the 2-groups listed in Theorem 3.1, we get a contradiction. Assume that $H\cong L_3(4)$. Then |P| = 2^6, and Z(P) $\cong$ $Z_2 \times Z_2$. Checking the 2-groups listed in Theorem 3.3, we get a contradiction. Assume that H $\cong$ $L_2(p)$, where p is a prime and p = 2^m + 1 or 2^m − 1. Then P is a dihedral group of order 2^m (see [3, ChII, Theorem 8.27]). Checking the 2-groups listed in Theorem 3.3, we conclude that P $\cong$ $Z_2 \times Z_2$ or D_8. Thus either p = |P| + 1 = 5 and then H $\cong$ $L_2(5)$, or p = |P| − 1 = 7 and then H $\cong$ $L_2(7)$.

Claim 4. G = H $\cong$ $L_2(q)$, where q = 5, 7, 9. It suffices to show that G = H. Otherwise, H < G $\leq$ Aut(H). We will apply [1] to get a contradiction. Assume that H $\cong$ A_5 (or $L_2(7)$). Then G $\cong$ S_5 (or PGL(2, 7)) has an element of order 6, so 2, 3 lie in the same prime component of G. However neither S_5 nor PGL(2, 7) has a nilpotent Hall {2, 3}-subgroup, a contradiction. Assume that H $\cong$ $L_2(9)$. Then G contains a subgroup which is isomorphic to $L_2(9) : 2_1$, $L_2(9) : 2_2$ or $L_2(9) : 2_3$ (see [1]). If $L_2(9) : 2_1\leq G$, then G has an element of order 6 but has no nilpotent Hall {2, 3}- subgroup, a contradiction. If $L_2(9) : 2_2 \leq G$, then G has an element of order 10 but has no nilpotent Hall {2, 5}-subgroup, a contradiction. If $L_2(9) : 2_3 \leq G$, then a Sylow 2-subgroup U of $L_2(9) : 2_3$ has order 16 and |Z(U)| = 2, and we also get a contradiction by checking the 2-groups listed in Theorem 3.3. Thus G = H as desired.

Conclusions

We conclude this paper by asking two important questions: Let H be a subgroup of a finite group G. Clearly, $H \geq H \cap H^x \geq H_G =: \cap_{x\in G} H^x \geq 1$. We call H is a CTI-subgroup of G if $H \cap H^x = H$ or H_G for any x $\in$ G. Our question is to classify the finite p-groups (or finite groups) all of whose subgroups (or abelian subgroups) are CTI-subgroups. Secondly, what can we say about the finite groups with no nontrivial TI-subgroup. Here a trivial TI-subgroup is a normal subgroup or a subgroup of prime order.

Conflicts of interest

Authors declare no conflict of interest.

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