Math 2471 - Calc 3

Line Integrals

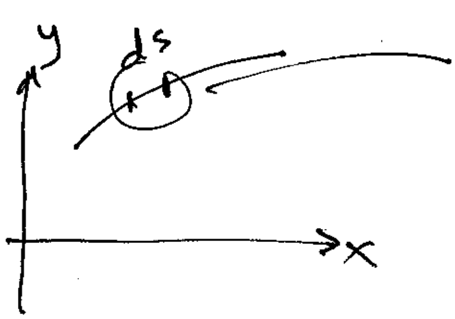
Consider that we have a wire with a variable density $\rho(x,y)$ that changes with respect to x & y then the total mass

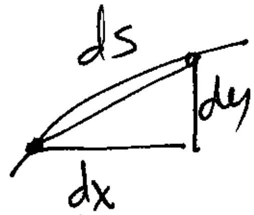
$$\int_C \rho(x,y)\,ds$$

where c is the curve of the wire &

ds - arc length.

From Calc I a Calc 2





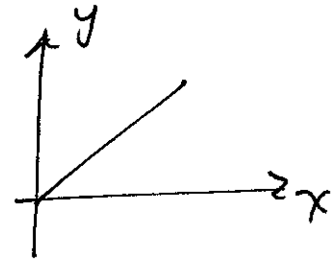
$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2 + \left(\frac{dy}{dx}\right)^2 (dx)^2}$$

$$= \sqrt{1 + y'^2}\,dx$$

so $\int_C f(x,y)\,ds = \int_a^b f(x,g(x))\sqrt{1+g'^2}\,dx$

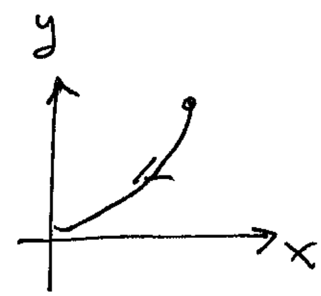
Ex Find $\int_C (x^2+y)\,ds$ along $y=x$ from $x=0,1$

so $ds = \sqrt{1+y'^2}\,dx = \sqrt{1+1}\,dx$



$\int_0^1 (x^2+x)\sqrt{2}\,dx$

$\sqrt{2}\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\Big|_0^1 = \sqrt{2}\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{5\sqrt{2}}{6}$



Same problem but along $y=x^2$

from $(1,1)\to(0,0)$

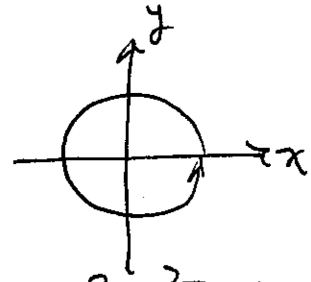
so $y=x^2 \quad y'=2x$

$\int_1^0 (x^2+x^2)\sqrt{1+4x^2}\,dx = -\int_0^1 2x^2\sqrt{1+4x^2}\,dx$

-1.2127

Ex Evaluate

$$\int_C x\,ds \quad \text{where } c \text{ is the circle ccw}$$



$x^2+y^2=1$

So we would have

$y = \pm\sqrt{1-x^2}$ & $1+y'^2$ would be complicated

its easier to parametrize the circle

$x = \cos t,\ y = \sin t \qquad 0 \le t \le 2\pi$

so let's convert every thing into t

Now $\quad ds = \sqrt{(dx)^2 + (dy)^2}$

$$= \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right](dt)^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

So $\displaystyle\int_C x\,ds = \int_{t_1}^{t_2} x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Now $x = \cos t,\ y = \sin t$

$$\frac{dx}{dt} = -\sin t \quad \frac{dy}{dt} = \cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t + \cos^2 t = 1$$

so $$\int_0^{2\pi} \cos t \, 1 \, dt = +\sin t \Big|_0^{2\pi} = 0$$

How about line integrals in 3-D

$$\int_C f(x,y,z)\,ds$$

so now the curve is in 3-D

so if $x = x(t),\ y = y(t)\ \ z = z(t)$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Ex $\int_C xyz\,ds \qquad \bar{r}(t) = \langle \cos t, \sin t, 3t \rangle$

$0 \le t \le 4\pi$

so $x = \cos t,\ y = \sin t,\ z = 3t$

$\frac{dx}{dt} = -\sin t \quad \frac{dy}{dt} = \cos t \quad \frac{dz}{dt} = 3$

$ds = \sqrt{\cos^2 t + \sin^2 t + 9}\,dt = \sqrt{10}$

$\int_0^{4\pi} \cos t \sin t \cdot 3t \sqrt{10}\,dt$

$3\sqrt{10} \int_0^{4\pi} t \sin t \cos t\,dt$

$\frac{3\sqrt{10}}{2} \int_0^{4\pi} t \sin 2t\,dt \quad \leftarrow$ parts

$= \frac{3\sqrt{10}}{2}\left(\frac{1}{4}\sin 2t - \frac{t}{2}\cos 2t \right)\Big|_0^{4\pi}$

$= -3\sqrt{10}\pi.$

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#15, 17, 25, 27, 29