Math 6345 - Advanced ODEs

Elementary ODE Review

1 Second Order Equations

Ordinary differential equations of the form

$$y'' = F(x, y, y') \tag{1}$$

are called second order ordinary differential equations. We usually only consider linear equations. These are of the form

$$a(x)y'' + b(x)y' + c(x)y = f(x), \tag{2}$$

or, dividing by $a(x)$ gives

$$y'' + p(x)y' + q(x)y = f(x), \tag{3}$$

noting that we have called $f/a = f$. If $f = 0$, we say the equation is "homogeneous", otherwise, it is "nonhomogeneous."

1.1 Reduction of Order

Given a second order homogeneous equation equation like (3), if we know one solution, say $y = y_0(x)$, then it is possible to reduce the order of the original equation using the substitution $y = uy_0$. The following example demonstrates. Consider

$$(x-1)y'' - xy' + y = 0.$$

We see that one solution is $y = e^x$. If we let $y = e^x u$, then substituting all derivatives and expanding, we obtain

$$(x-1)u'' + (x-2)u' = 0.$$

If we let $u' = v$, then

$$(x-1)v' + (x-2)v = 0,$$

a separable ODE. Integrating gives

$$v = (x-1)e^{-x},$$

and since $u' = v$, we integrate further gives

$$u = -xe^{-x}.$$

Therefore,

$$y = e^x u = e^x(-xe^{-x}) = -x.$$

As we can suppress the negative sign, we obtain

$$y_2 = x,$$

as the second independent solution. To show this, we consider the Wronskian. Recall that the Wronskian is defined by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}.$$

Here, we see that $W = (x-1)e^x \neq 0 \ \forall \ x$. Thus, the solutions are independent. The general solution is

$$y = c_1 x + c_2 e^x.$$

1.2 Constant Coefficient Equations

If the coefficients in (2) (with $f = 0$) are constant, we then have

$$ay'' + by' + cy = 0. \tag{4}$$

Motivated by first order constant coefficient equations

$$y' - my = 0,$$

where the solution $y = e^{mx}$, we seek solutions of this form for second order equations. Substitution into (4) gives

$$\left(am^2 + bm + c\right)e^{mx} = 0 \tag{5}$$

from which we deduce that

$$am^2 + bm + c = 0. \tag{6}$$

This is called the "characteristic equation" since it characterizes the solution. Solutions for m gives rise to three cases:

(i) two real distinct roots,

(ii) two real repeated roots,

(iii) two complex roots.

1.2.1 Two real distinct roots

If the roots are $m = r_1$ and $m = r_2$, then the solution is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

1.2.2 Two real repeated roots

If both roots are $m = r$, then the solution is

$$y = c_1 e^{rx} + c_2 x e^{rx}.$$

1.2.3 Two complex roots

If the roots are $m = \alpha \pm \beta i$, then the solution is

$$y = c_1 e^{\alpha x} \sin \beta x + c_2 e^{\alpha x} \cos \beta x.$$

Example 1

$$y'' - 5y' + 6y = 0.$$

Here, the characteristic equation is

$$m^2 - 5m + 6 = 0.$$

Thus, the roots are $m = 2$ and $m = 3$ and the solution is

$$y = c_1 e^{2x} + c_2 e^{3x}.$$

Example 2

$$y'' - 6y' + 9y = 0.$$

Here, the characteristic equation is

$$m^2 - 6m + 9 = 0.$$

Here, the roots are $m = 3$ and $m = 3$, (repeated). Thus, the solution is

$$y = c_1 e^{3x} + c_2 x e^{3x}.$$

Example 3

$$y'' + 4y' + 13y = 0.$$

Here, the characteristic equation is

$$m^2 + 4m + 13 = 0.$$

The roots are $m = -2 \pm 3i$, and the solution is

$$y = c_1 e^{-2x} \sin 3x + c_2 e^{-2x} \cos 3x.$$

1.3 Cauchy-Euler Equations

Equations of the form

$$ax^2 y'' + bxy' + cy = 0, \tag{7}$$

are called Cauchy-Euler equation. Motivated by its first order equivalent

$$xy' - my = 0 \tag{8}$$

who has solution $y = x^m$, we could seek solutions in this form (and we did). However, under the transformation $x = e^t$, the derivatives transform as

$$x\frac{d}{dx} = \frac{d}{dt}, \quad x^2 \frac{d^2}{dx^2} = \frac{d^2}{dt^2} - \frac{d}{dt} \tag{9}$$

and (7) transforms to

$$a\ddot{y} + (b - a)\dot{y} + cy = 0, \tag{10}$$

a constant coefficient 2nd order ODE! Thus, the results from the previous section apply.

1.4 Nonhomogeneous Equations

To solve equation (3), we solve this in two parts. We first solve the complementary equation

$$y'' + p(x)y' + q(x)y = 0,$$

which gives rise to $y = y_c(x)$, then we find a particular solution $y = y_p(x)$ to

$$y'' + p(x)y' + q(x)y = f(x).$$

The general solution is then given by

$$y = y_c + y_p.$$

To find a particular solution, we use the "variation of parameters" technique. From the complementary solution we have

$$y = c_1 y_1 + c_2 y_2.$$

We replace the constants with parameters u and v and substitute into the original equation, *i.e.*

$$y = uy_1 + vy_2. \tag{11}$$

As there are two unknowns, u and v, we have some flexibility. After taking the first derivative of (11)

$$y' = u'y_1 + uy'_1 + v'y_2 + vy'_2 \tag{12}$$

we set

$$u'y_1 + v'y_2 = 0, \tag{13}$$

leaving

$$y' = uy'_1 + vy'_2. \tag{14}$$

Further, substitution of this and the second derivative into the original equation leads to

$$u'y_1 + v'y'_2 = f(x) \tag{15}$$

from which we can solve (13) and (15) for the unknowns for u' and v'. After integrating u' and v', we obtain the particular solution from (11) . The following example illustrates.

Example 4

$$y'' + y = \sec x. \tag{16}$$

The solution of the homogeneous problem

$$y'' + y = 0,$$

is

$$y = c_1 \sin x + c_2 \cos x.$$

Thus, we look for a particular solution in the form

$$y = u \sin x + v \cos x. \tag{17}$$

Taking the first derivative gives

$$y' = u' \sin x + u \cos x + v' \cos x - v \sin x, \tag{18}$$

on which we set

$$u' \sin x + v' \cos x = 0. \tag{19}$$

This leaves

$$y' = u \cos x - v \sin x. \tag{20}$$

The second derivative gives

$$y'' = u' \cos x - u \sin x - v' \sin x - v \cos x. \tag{21}$$

Substituting (17), (20) and (21) into (16) and simplifying gives

$$u' \cos x - v' \sin x = \sec x. \tag{22}$$

Solving (19) and (22) for u' and v' gives

$$u' = 1, \quad v' = -\tan x.$$

These are easily solved giving

$$u = x, \quad v = \ln|\cos x| \tag{23}$$

and thus, from (17) and (23) we obtain the particular solution

$$y_p = x \sin x + \cos x \ln|\cos x|. \tag{24}$$

The general solution is then given by

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 \sin x + c_2 \cos x + x \sin x + \cos x \ln|\cos x|. \end{aligned} \tag{25}$$