Math 4315 - PDEs Sample Test 3 Solutions

1. Determine the Fourier series for

(i)

$$f(x) = \begin{cases} 1 & \text{if } -2 \leq x < 0 \\ x+1 & \text{if } \quad 0 \leq x \leq 2 \end{cases}$$

(ii)

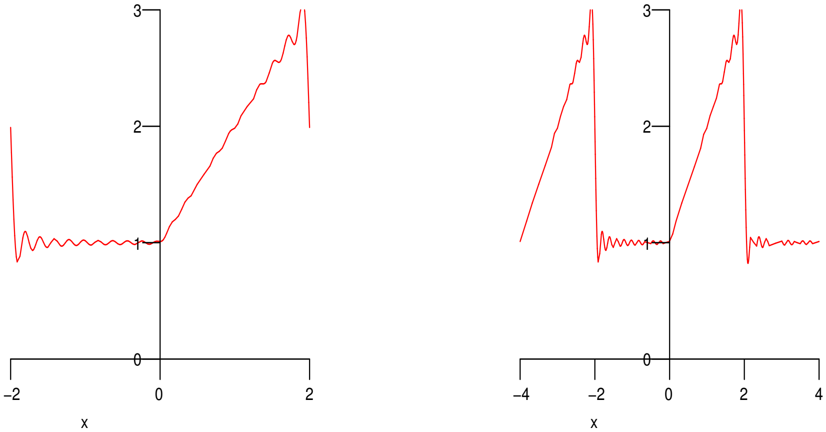
$$f(x) = \begin{cases} -x & \text{if } -1 \leq x < 0 \\ x^2 & \text{if } \quad 0 \leq x \leq 1 \end{cases}$$

Solution 1i

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-2}^{0} 1\, dx + \frac{1}{2}\int_{0}^{2} (x+1)\, dx = 3 \\ a_n &= \frac{1}{2}\int_{-2}^{0} \cos\frac{n\pi x}{2}\, dx + \frac{1}{2}\int_{0}^{2} (x+1)\cos\frac{n\pi x}{2}\, dx = \frac{2(\cos n\pi - 1)}{n^2\pi^2} \\ b_n &= \frac{1}{2}\int_{-2}^{0} \sin\frac{n\pi x}{2}\, dx + \frac{1}{2}\int_{0}^{2} (x+1)\sin\frac{n\pi x}{2}\, dx = -\frac{2\cos n\pi}{n\pi} \end{aligned}$$

The solution is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{i=1}^{\infty} a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{2} \\ &= \frac{3}{2} + \sum_{i=1}^{\infty} \frac{2(\cos n\pi - 1)}{n^2\pi^2}\cos\frac{n\pi x}{2} - \frac{2\cos n\pi}{n\pi}\sin\frac{n\pi x}{2} \end{aligned}$$

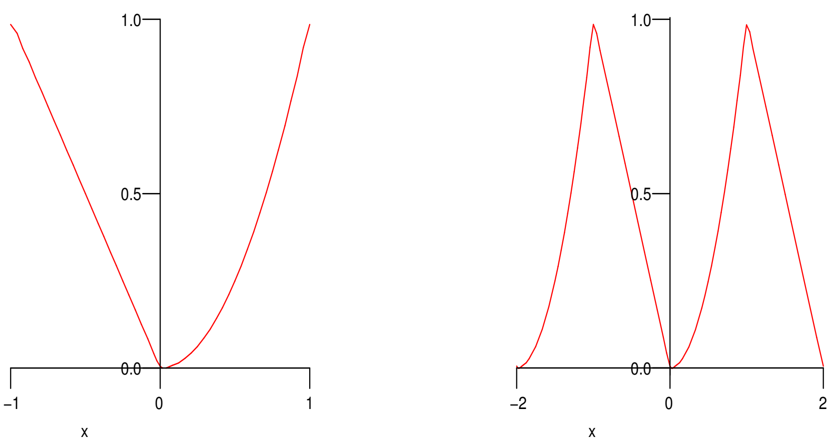


Solution 1ii

$$
\begin{aligned}
a_0 &= \int_{-1}^{0} -x\,dx + \int_0^1 x^2\,dx = \frac{5}{6} \\
a_n &= \int_{-1}^{0} -x\cos n\pi x\,dx + \int_0^1 x^2\cos n\pi x\,dx = \frac{3\cos n\pi - 1}{n^2\pi^2} \\
b_n &= \int_{-1}^{0} -x\sin n\pi x\,dx + \int_0^1 x^2\sin n\pi x\,dx = \frac{2(\cos n\pi - 1)}{n^3\pi^3}
\end{aligned}
$$

The solution is

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{i=1}^{\infty} a_n\cos n\pi x + b_n\sin n\pi x \\
&= \frac{5}{12} + \sum_{i=1}^{\infty} \frac{3\cos n\pi - 1}{n^2\pi^2}\cos n\pi x + \frac{2(\cos n\pi - 1)}{n^3\pi^3}\sin n\pi x.
\end{aligned}
$$



2. Solve

$$u_t = u_{xx}, \qquad 0 < x < L,$$

subject to the initial condition and boundary conditions

$$(i) \quad u(x,0) = 5x - x^2, \quad u(0,t) = 0, \quad u(4,t) = 4$$

$$(ii) \quad u(x,0) = \begin{cases} x^2 + 1 & \text{if } 0 < x < 1, \\ 2(x-2)^2 & \text{if } 1 < x < 2. \end{cases}, \quad u(0,t) = 1, \quad u(2,t) = 0$$

Solution 2i

Before we can use separation of variables, it is necessary to transform this problem to one that has fixed zero boundary conditions. If we let

$$u = v + ax + b,$$

imposing the boundary conditions gives

$$\begin{aligned} u(0,t) &= v(0,t) + a \cdot 0 + b, \\ u(4,t) &= v(4,t) + a \cdot 4 + b, \end{aligned}$$

and substituting the actual BCs and the desired ones gives

$$\begin{gathered} 0 = 0 + b, \\ 4 = 0 + 4a + b, \end{gathered}$$

giving $a = 1$ and $b = 0$. We now consider the IC

$$u(x,0) = v(x,0) + x,$$

giving

$$v(x,0) = 4x - x^2.$$

Thus, the new problem is

$$\begin{gathered} v_t = v_{xx}, \quad 0 < x < L, \\ v(x,0) = 4x - x^2, \quad v(0,t) = 0, \quad v(4,t) = 0. \end{gathered}$$

If we assume separable solutions in the form $v = X(x)T(t)$, then PDE separates giving

$$\frac{T'}{T} = \frac{X''}{X},$$

from which we obtain

$$T' = \lambda T, \quad X'' = \lambda X.$$

The boundary conditions become $X(0) = 0, \quad X(4) = 0$. The solution of the X equation is

$$X = c_1 \sin kx + c_2 \cos kx,$$

where $\lambda = -k^2$. To satisfy both BCs we must choose $k = \frac{n\pi}{4}$ and $c_2 = 0$. This then gives

$$X = c_1 \sin \frac{n\pi}{4} x.$$

Solving for T gives

$$T = c_3 e^{-\frac{n^2\pi^2}{16} t}$$

which in turn gives

$$v = \sum_{n=1}^{\infty} b_n \, e^{-\frac{n^2\pi^2}{16} t} \sin \frac{n\pi}{4} x,$$

where we have taken $b_n = c_1 c_3$. Imposing the initial condition gives

$$b_n = \frac{2}{4} \int_0^4 (4x - x^2) \sin \frac{n\pi}{4} x \, dx = \frac{64(1 - \cos n\pi)}{n^3\pi^3}.$$

This then gives the solution as

$$v = \sum_{n=1}^{\infty} \frac{64(1 - \cos n\pi)}{n^3\pi^3} \, e^{-\frac{n^2\pi^2}{16} t} \sin \frac{n\pi}{4} x,$$

and u is

$$u = x + \sum_{n=1}^{\infty} \frac{64(1 - \cos n\pi)}{n^3\pi^3} \, e^{-\frac{n^2\pi^2}{16} t} \sin \frac{n\pi}{4} x,$$

Solution 2ii

Before we can use separation of variables, it is necessary to transform this problem to one that has fixed zero boundary conditions. If we let

$$u = v + ax + b,$$

we find that choosing $a = -1/2$ and $b = 1$ given the new problem to solve

$$v_t = v_{xx}, \quad 0 < x < 2,$$

$$v(x,0) = \begin{cases} x^2 + \frac{x}{2} & \text{if } 0 < x < 1, \\ 2x^2 - \frac{15}{2}x + 7 & \text{if } 1 < x < 2. \end{cases}, \quad v(0,t) = 0, \quad v(2,t) = 0.$$

If we assume separable solutions in the form $u = X(x)T(t)$, then PDE separates giving

$$\frac{T'}{T} = \frac{X''}{X},$$

from which we obtain

$$T' = \lambda T, \quad X'' = \lambda X.$$

The boundary conditions become $X(0) = 0, \quad X(2) = 0$. The solution of the X equation is

$$X = c_1 \sin kx + c_2 \cos kx,$$

where $\lambda = -k^2$. To satisfy both BCs we must choose $\lambda = \frac{n^2\pi^2}{4}$ and $c_2 = 0$. This then gives

$$X = c_1 \sin \frac{n\pi}{2} x.$$

Solving for T gives

$$T = c_3 e^{-\frac{n^2\pi^2}{4} t}$$

which in turn gives

$$u = \sum_{n=1}^{\infty} b_n \, e^{-\frac{n^2\pi^2}{4} t} \sin \frac{n\pi}{2} x,$$

where we have taken $b_n = c_1 c_3$. Imposing the initial condition gives

$$\begin{aligned} b_n &= \frac{2}{2} \int_0^1 (x^2 - \frac{x}{2}) \sin \frac{n\pi}{2} x \, dx + \frac{2}{2} \int_1^2 (2x^2 - \frac{15x}{2} + 7) \sin \frac{n\pi}{2} x \, dx \\ &= \frac{(-16 - 16\cos\frac{n\pi}{2} + 32\cos n\pi)}{n^3\pi^3} + \frac{24 \sin \frac{n\pi}{2}}{n^2\pi^2}. \end{aligned}$$

This then gives v as

$$v = \sum_{n=1}^{\infty} b_n \, e^{-\frac{n^2\pi^2}{4} t} \sin \frac{n\pi}{2} x,$$

and u as

$$u = -\frac{1}{2} x + 1 + \sum_{n=1}^{\infty} b_n \, e^{-\frac{n^2\pi^2}{4} t} \sin \frac{n\pi}{2} x,$$

3. Solve Laplace's equation

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < L,$$

subject to the boundary conditions

$$(i) \quad u(x,0) = 0, \quad u(0,y) = 0, \quad u(x,1) = x^2, \quad u(1,y) = 0,$$

$$(ii) \quad u(x,0) = 0, \quad u(0,y) = 0, \quad u(x,2) = 0, \quad u(2,y) = 2y - y^2.$$

Solution 3i

If we assume separable solutions of the form

$$u(x,y) = X(x)Y(y),$$

then

$$X''Y + XY'' = 0.$$

or

$$\frac{X''}{X} + \frac{Y''}{Y} = 0,$$

This gives

$$\frac{X''}{X} = \lambda, \quad \frac{Y''}{Y} = -\lambda, \quad \lambda \quad \text{constant.}$$

The boundary conditions become

$$X(0) = 0, \quad X(1) = 0, \quad Y(0) = 0.$$

In order to satisfy the X BCs, we need $\lambda = -k^2$ and so solving for X gives

$$X = c_1 \sin kx + c_2 \cos kx.$$

The X boundary conditions gives $k = n\pi, \;\; k \in \mathbb{Z}^+$ and $c_2 = 0$ so

$$X(x) = c_1 \sin n\pi x,$$

and further

$$Y(y) = c_3 \sinh n\pi y + c_4 \cosh n\pi y.$$

Since $Y(0) = 0$ this implies $c_4 = 0$ so

$$u = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi y. \quad (a_n = c_1 c_3)$$

From the last boundary condition

$$u(x,1) = x^2 = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi,$$

If $A_n = a_n \sinh n\pi$, then

$$A_n = \frac{2}{1}\int_0^1 x^2 \sin n\pi x \, dx = \frac{4(\cos n\pi - 1)}{n^3\pi^3} - 2\frac{\cos n\pi}{n\pi}.$$

Thus, the solution is

$$u(x,y) = \sum_{n=1}^{\infty} \left(\frac{4(\cos n\pi - 1)}{n^3\pi^3} - 2\frac{\cos n\pi}{n\pi} \right) \sin n\pi x \frac{\sinh n\pi y}{\sinh n\pi}.$$

Solution 3ii

If we assume separable solutions of the form

$$u(x,y) = X(x)Y(y),$$

then

$$X''Y + XY'' = 0.$$

or

$$\frac{X''}{X} + \frac{Y''}{Y} = 0,$$

This gives

$$\frac{X''}{X} = \lambda, \qquad \frac{Y''}{Y} = -\lambda, \quad \lambda \quad \text{constant.}$$

The boundary conditions become

$$X(0) = 0, \quad Y(0) = 0, \quad Y(2) = 0.$$

In order to satisfy the Y BCs, we need $\lambda = k^2$ and so solving for Y gives

$$Y = c_1 \sin ky + c_2 \cos ky.$$

The Y boundary conditions gives $k = \frac{n\pi}{2}, \;\; k \in \mathbb{Z}^+$ and $c_2 = 0$ so

$$Y(y) = c_1 \sin \frac{n\pi}{2} y,$$

and further

$$X(x) = c_3 \sinh \frac{n\pi}{2} x + c_4 \cosh \frac{n\pi}{2} x.$$

Since $X(0) = 0$ this implies $c_4 = 0$ so

$$u = \sum_{n=1}^{\infty} a_n \sinh \frac{n\pi}{2} x \sin \frac{n\pi}{2} y. \quad (a_n = c_1 c_3)$$

From the last boundary condition

$$u(2, y) = 2y - y^2 = \sum_{n=1}^{\infty} a_n \sinh n\pi \sin \frac{n\pi}{2} y,$$

If $A_n = a_n \sinh n\pi$, then

$$A_n = \frac{2}{2} \int_0^2 (2y - y^2) \sin \frac{n\pi}{2} y \, dy = \frac{16(1 - \cos n\pi)}{n^3 \pi^3}.$$

Thus, the solution is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{16(1 - \cos n\pi)}{n^3 \pi^3} \frac{\sinh \frac{n\pi}{2} x}{\sinh n\pi} \sin \frac{n\pi}{2} y.$$