LSWAVE: a MATLAB software for the least-squares wavelet and cross-wavelet analyses

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Abstract

The least-squares wavelet analysis (LSWA) is a robust method of analyzing any type of time/data series without the need for editing and preprocessing of the original series. The LSWA can rigorously analyze any non-stationary and equally/unequally spaced series with an associated covariance matrix that may have trends and/or datum shifts. The least-squares cross-wavelet analysis complements the LSWA in the study of the coherency and phase differences of two series of any type. A MATLAB software package including a graphical user interface is developed for these methods to aid researchers in analyzing pairs of series. The package also includes the least-squares spectral analysis, the antileakage least-squares spectral analysis, and the least-squares cross-spectral analysis to further help researchers study the components of interest in a series. We demonstrate the steps that users need to take for a successful analysis using three examples: two synthetic time series, and a Global Positioning System time series.

Keywords Least-squares spectral analysis, Antileakage least-squares spectral analysis, GPS time series analysis, Least-squares wavelet analysis, Least-squares cross-spectral analysis, Least-squares cross-wavelet analysis

Introduction

Vaníček (1969) proposed the least-squares spectral analysis (LSSA) to analyze unequally spaced time series. The LSSA estimates the spectrum based on the least-squares fit of sinusoids of specified frequencies to the entire series. Pagiatakis (1999) studied the statistical properties of the least-squares spectrum and defined critical values at specific confidence levels to identify stochastically significant peaks in the spectrum.

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Ghaderpour et al. (2018b) proposed an iterative method based on the LSSA namely, the antileakage least-squares spectral analysis (ALLSSA) that uses a preselected set of frequencies to accurately estimate the statistically significant spectral peaks corresponding to the wide-sense stationary components of a series. After simultaneously suppressing several significant spectral peaks with the highest power/energy, an iterative process is performed to reoptimize the estimated frequencies, reducing the computational cost. They showed that the ALLSSA performs better than the antileakage Fourier transform (Xu et al. 2005), and interpolation by matching pursuit (Vassallo et al. 2010).

Ghaderpour and Pagiatakis (2017) developed a new method of spectral analysis, namely, the least-squares wavelet analysis (LSWA), that decomposes a time series into the time-frequency domain, allowing the detection of short-duration signatures in the series. The LSWA simultaneously considers the correlations between the sinusoidal functions and constituents of known forms, such as datum shifts, trends, and any correlated noise. The LSWA computes spectrograms rigorously for equally or unequally spaced series without any preprocessing, modification or editing. The stochastic surfaces defined for the spectrogram show the significant spectral peaks at a certain confidence level that is usually 95% or 99%. Ghaderpour and Pagiatakis (2017) showed the robust performance of the LSWA compared to the state-of-the-art time series analysis methods, such as the continuous wavelet transform (Daubechies 1990), weighted wavelet Z-transform (Foster 1996), Hilbert-Huang transform (Huang and Wu 2008) and constrained least-squares spectral analysis (Puryear et al. 2012).

Pagiatakis et al. (2007) proposed the least-squares self-coherency analysis to analyze two series together, based on the LSSA. Ghaderpour et al. (2018a) proposed an alternative method, namely, the least-squares cross-spectral analysis (LSCSA), to compute the cross-spectrum of two series. The LSCSA obtains a cross-spectrum by multiplying the least-squares spectra of the two series, an appropriate coherence analysis of wide-sense stationary series. Ghaderpour et al. (2018a) also developed the least-squares cross-wavelet analysis (LSCWA), a novel method of analyzing two series, based on the LSWA. The LSCWA shows how much constituents with common frequencies in both series are coherent and whether the coherency is significant at a certain confidence level using rigorous statistical inference. In the LSCWA, the series do not have to be equally spaced and equally weighted, and they do not need to have the same sampling rate. The LSCWA does not require any preprocessing or editing of the original series, superseding the state-of-the-art methods, such as the cross-spectral analysis (von Storch and Zwiers 2001) and cross-wavelet transform (Torrence and Compo 1998).

Ghaderpour (2018) illustrated the ALLSSA and LSWA algorithms with detailed flowcharts in Chapters 3 and 6, respectively. These algorithms can be used with any series. The least-squares based methods, namely, the LSSA, ALLSSA, LSWA, LSCSA, and LSCWA, have been all programmed in MATLAB. A user-friendly graphical user interface (GUI) is also designed for these methods. We briefly describe our software features and then using synthetic and real examples, we demonstrate how one can appropriately analyze any series.

Software features and spectral analysis

The spectral analysis methods presented here are based on Ghaderpour and Pagiatakis (2017) and Ghaderpour et al. (2018a, b), using the same symbols. The main purpose of these methods is studying the periodicities of constituents, coherency, and hidden signatures in any series. For example, accurate estimation of frequencies, amplitudes, and phases of components in Global Positioning System (GPS) time series are often challenging because these series are often unequally spaced, unequally weighted, non-stationary, and present datum shifts and trends.

Software Inputs

The main input data sets are column vectors for the times and series values along with the associated covariance matrices of the series values if they exist. A set of frequencies $\mathbf{\Omega} = \{\omega_k;\ k = 1, 2, \ldots, \kappa\}$ (cycles per unit time) defining the spectrum bandwidth of interest may also be entered that can be any set of positive real numbers based on the scope of analysis. A lower bound of this set may be the inverse of twice of the series length under consideration with upper bound $M/2$, where M is the average sampling rate.

In the LSWA and LSCWA, the translating (shifting) window size parameters, M, L_1, and L_0 may also be entered. Parameter M determines the Nyquist frequency when series is equally spaced. Since the Nyquist frequency is not explicitly defined for inherently unequally spaced series, $M/2$ may be selected as an upper bound for the frequency band. Craymer (1998) showed how to explicitly determine the Nyquist frequency for series that are equally spaced as multiples of some common interval or equally spaced with gaps.

Parameter L_1 defines the number of cycles of sinusoids within the translating windows, and L_0 is an additional number of samples within the translating windows in the least-squares fitting process. These parameters determine the segment size, $L = \lfloor L_1 M/\omega_k \rfloor + L_0$, where $\lfloor \cdot \rfloor$ is the floor function. If $L_1 > 0$, the window size decreases when the frequency increases, allowing the detection of short-duration signals. In unequally spaced series, the sinusoids of frequency ω_k within the translating windows may not exactly complete L_1 cycles, and so L_1 is approximate. The larger the L_0, the higher the frequency resolution but the poorer the time resolution will be in the spectrogram. The GUI sets $L_1 = 2$ and $L_0 = 20$ as default.

Foster (1996) recommended an effective scale for the Morlet wavelet in the least-squares sense, that is 0.0125. By selecting this scale, the weight matrix within each translating window will be a diagonal matrix whose diagonal elements are the Gaussian values. Ghaderpour and Pagiatakis (2017) showed that this selection smooths the spectral peaks in the spectrogram with an optimal time-frequency resolution.

When the series values have been derived from populations of random variables following multidimensional normal distributions, one can rigorously identify the statistically significant peaks in the spectrum, cross-spectrum, spectrogram, and cross-spectrogram at a certain confidence level. The critical value increases when the window size decreases and vice versa.

Series may have datum shifts or jumps. For example, in GPS time series, shifts in position are mainly caused by a change in the antenna reference point or tectonic displacement. Rodionov (2004) proposed a sequential algorithm for early detection of datum shifts. It is recommended that users enter the indices of the start times of datum shifts prior to the analysis. Since the LSWA is a segment-wise algorithm, it is not as sensitive as the LSSA or ALLSSA to datum shifts. The software has an option to fit and remove a polynomial of degree three or less from each segment of series, being considered simultaneously with sinusoids.

In certain experiments, there are some constituents of known frequencies that contaminate the series. The software can remove them simultaneously with the trends to study the residual segments for any hidden signatures, coherency, and phase differences. These constituents can also be estimated by analyzing the original series, so they will be known to users for the next round of analysis. There is an option in the GUI to choose the number of phase arrows being plotted toward time and frequency axes in the LSCWA. Ghaderpour et al. (2018a) showed that the direction of an arrow indicates how much the constituent of the second series segment lags or leads the constituent of the first series segment

Software Outputs

The main outputs from the analysis of a single series are the spectrum, antileakage spectrum, and spectrogram; and from the two series analysis, the cross-spectrum and cross-spectrogram with their critical values and phase information. Other outputs from the LSSA or ALLSSA are the estimated coefficients of the constituents of known forms and their estimated covariance matrix. The diagonal entries of this covariance matrix from top left to bottom right are the variances of estimated coefficients for datum shifts, trends, and cosine and sine functions of known frequencies/wavenumbers in ascending order. The GUI calculates the amplitudes and phases from the estimated coefficients of sinusoids with their errors (Appendix A). To display the stochastic surfaces, we use the 'freezeColors' tool developed by John Iversen that enables multiple colormaps (available from the Mathworks File Exchange).

Examples

A synthetic example

Two experimental time series are simulated in a controlled environment. Suppose that the first series is a voltage series simulated at 450 unequally spaced random times t_j's from zero to three hours using the MATLAB command *rand*, sorted in an ascending order. The voltages are:

$$f_1(t_j) = \begin{cases} h(t_j) + 15 & \text{if } 1 \le j < 101 \\ h(t_j) + 30 + 2\ \sin(60 \cdot 2\pi t_j) & \text{if } 101 \le j < 321 \\ h(t_j) + 10 & \text{if } 321 \le j \le 450 \end{cases} \tag{1}$$

where $h(t_j) = 4\sin(5 \cdot 2\pi t_j + \pi/6) + 2\sin(10 \cdot 2\pi t_j + 15\pi t_j^2) + 0.5\,\mathrm{wgn}(t_j)$ that contains a quadratic chirp signal, and MATLAB command 'wgn' is used to generate white Gaussian noise. This time series has two large datum shifts whose breaks are at $j = 101$ and $j = 321$ (Fig. 1, top panel).

Suppose that the second series is an ambient temperature series simulated at 600 unequally spaced random times t'_j's from zero to three hours. The temperature values in degrees Celsius (°C) are:

$$f_2(t'_j) = 4\sin\left(5 \cdot 2\pi t'_j + \frac{\pi}{3}\right) + \sin(35 \cdot 2\pi t'_j) + 0.5\,\mathrm{wgn}(t'_j) \tag{2}$$

where $1 \le \mathrm{j} \le 600$ and $0 \le \mathrm{t}'_\mathrm{j} \le 3$. The GUI calculates $\mathrm{M}_1 = 449/(\mathrm{t}_{450} - \mathrm{t}_1) \cong 150$ (samples/h) and M2 $= 599/(\mathrm{t}'_{600} - \mathrm{t}'_1) \cong 200$ (samples/h) and sets $\mathrm{L}_1 = 2$ cycles and $\mathrm{L}_0 = 20$ samples for both series. Using the average sampling rates M_1 and M_2, the GUI internally calculates the common upper bound for the frequency band as the minimum of $\mathrm{M}_1/2$

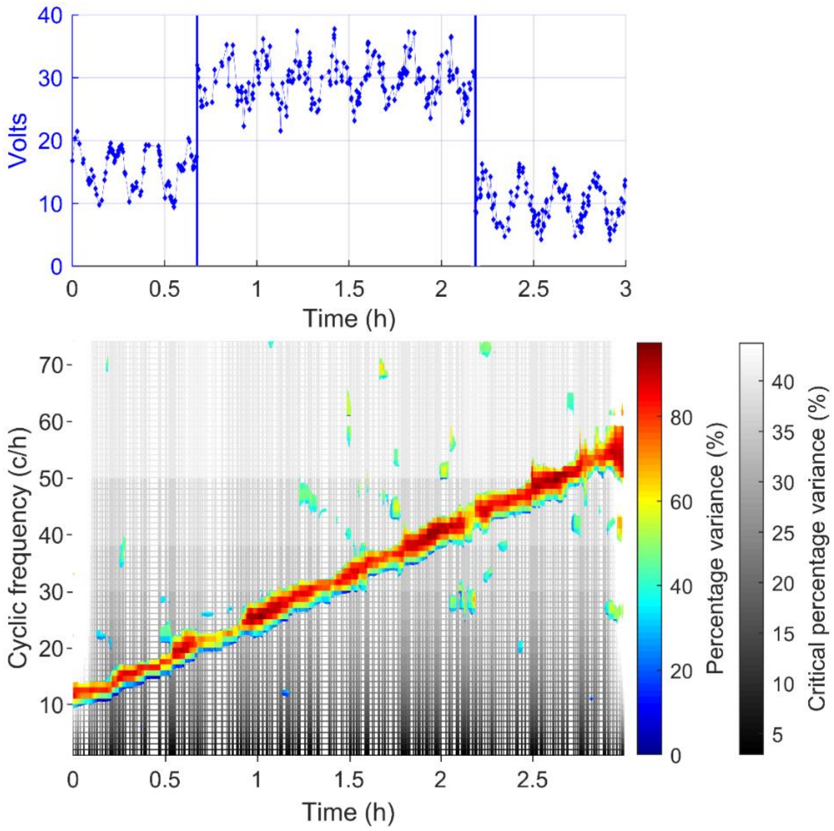

Fig. 1 A synthetic unequally spaced time series and its analysis result. The top panel shows the time series with two datum breaks shown by blue bars, and the bottom panel shows its residual spectrogram with its stochastic surface at 99% confidence level in gray

and $M_2/2$, that is approximately 75 c/h. We choose 74 equally spaced frequencies from 1 c/h to 75 c/h to generate an equally spaced spectrum at frequencies, $\boldsymbol{\Omega} = \{1,2,\ldots,74\}$ c/h. We also choose the significance level to be $\alpha = 0.01$ for a more robust determination of the spectral peaks.

Analysis of the first series

We enter the indices of datum shifts and apply the LSWA. After the LSWA detects the dominant sine wave of 5 c/h and amplitude 4 volts and the short duration sine wave 60 c/h and amplitude 2 volts, we suppress their spectral peaks from the spectrogram by entering their frequencies as known frequencies in a designed GUI panel and rerun the analysis. The bottom panel in Fig. 1 shows the result, the residual spectrogram. The dashed vertical lines in the spectrogram are displayed using the 'mesh' MATLAB command to show the series gaps. The spectrogram clearly shows the peaks corresponding to the quadratic chirp signal whose frequencies increase over time, unlike the spectrum not shown here.

Analysis of both series together

To study the coherency and phase differences between the components of the two series shown in the top panel of Fig. 2, we enter the datum breaks of the first series and apply the LSCWA. The bottom panel clearly shows the coherency between the sine wave of 5 c/h in the residual cross-spectrogram. The arrows on the cross-spectrogram show that the sine wave of 5 c/h in the temperature series leads the one in the voltage series about $\pi/6$ as we expected because $\pi/6 - \pi/3 = -\pi/6$. One may suppress the peaks at frequency 5 c/h in the cross-spectrogram to search for any other hidden coherency.

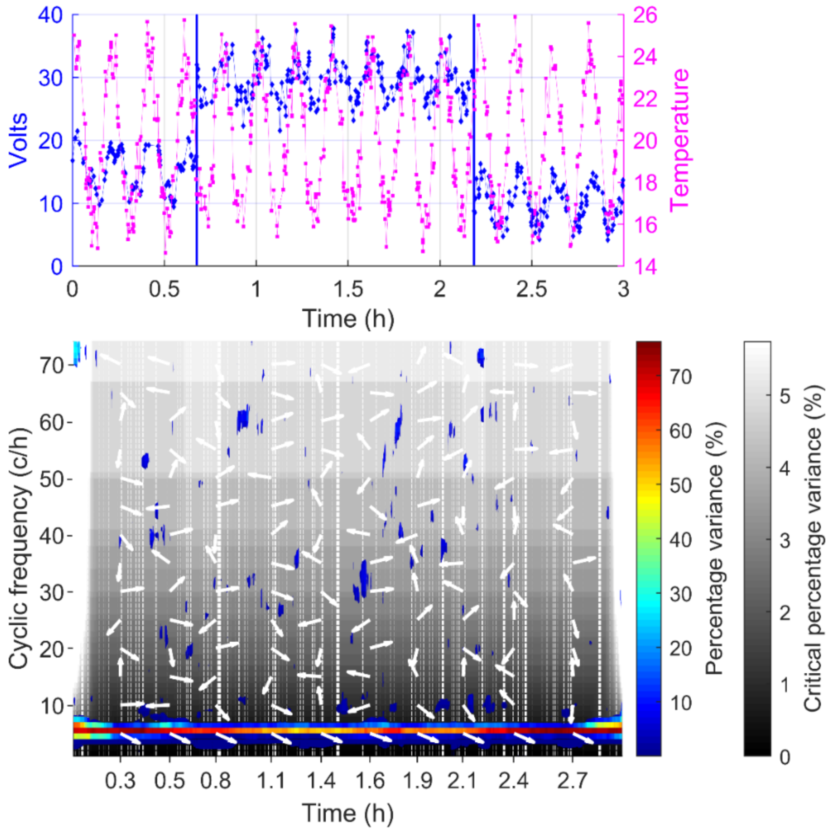

Fig. 2 Two synthetic time series and their coherency analysis. The first and second series are shown in blue and pink in the top panel, respectively. The bottom panel is their residual cross-spectrogram with stochastic surface at 99% confidence level in gray and phase arrows in white

Spectral analysis of a GPS time series

We analyze a GPS height time series from https://sideshow.jpl.nasa.gov/post/series.html. The station selected for this study is PRDS located in Priddis, Alberta, Canada. The top panel in Fig. 3 shows the time series, containing 5986 unequally spaced and unequally weighted samples. The average sampling rate is 358 samples/y or about 1 sample/d. To investigate if there are any annual, semi-annual, seasonal, and every two months components in the series, we choose $\boldsymbol{\Omega} = \{0.2, 0.4, \dots, 8\}$ c/y for the entire analysis.

We use the same estimated datum breaks posted in the website above that are 2002.9406, 2003.3841, 2006.6146, and 2012.5722 whose time indices are 304, 427, 1588, and 3737, respectively. Since in many cases, such as GPS time series, there is a consistent trend for all the series segments between offsets, the LSSA and ALLSSA are programmed to estimate the datum shifts and estimate one single slope for the entire series simultaneously. In the LSWA, the same process will be applied but within each frequency-dependent segment.

The bottom panel in Fig. 3 shows the least-squares spectrum after simultaneously removing the datum shifts and linear trend. The peaks approximately at frequencies 1, 2, and 4 c/y are statistically significant at 99% confidence level, correspond to the annual, semi-annual, and seasonal components in the series, respectively. A denser set of frequencies can be selected to estimate the signal frequencies more accurately.

Using set $\boldsymbol{\Omega}$, the ALLSSA will search around a small neighbourhood of significant peaks to estimate the signal frequencies more accurately in an iterative manner. Therefore, the estimated amplitudes for the annual and semi-annual components are 2.787 ± 0.097 mm and 1.217 ± 0.069 mm, respectively. These values are approximately in agreement with the ones posted in the website above that are 2.804 ± 0.9 mm and 1.226 ± 0.9 mm.

The frequencies of significant components with their corresponding amplitudes and total shifts of each segment in ascending order and the common slope are estimated and shown in Table 1. We also used another set of frequencies $\boldsymbol{\Omega} = \{0.2, 0.4, \dots, 200\}$ c/y and obtained the same results as listed in Table 1. The L2 norm of the original series and residual series are 498.28 mm and 398.20 mm, respectively. This low reduction of norm is due to random noise and other constituents, such as short duration signatures, daily periodic or aperiodic components.

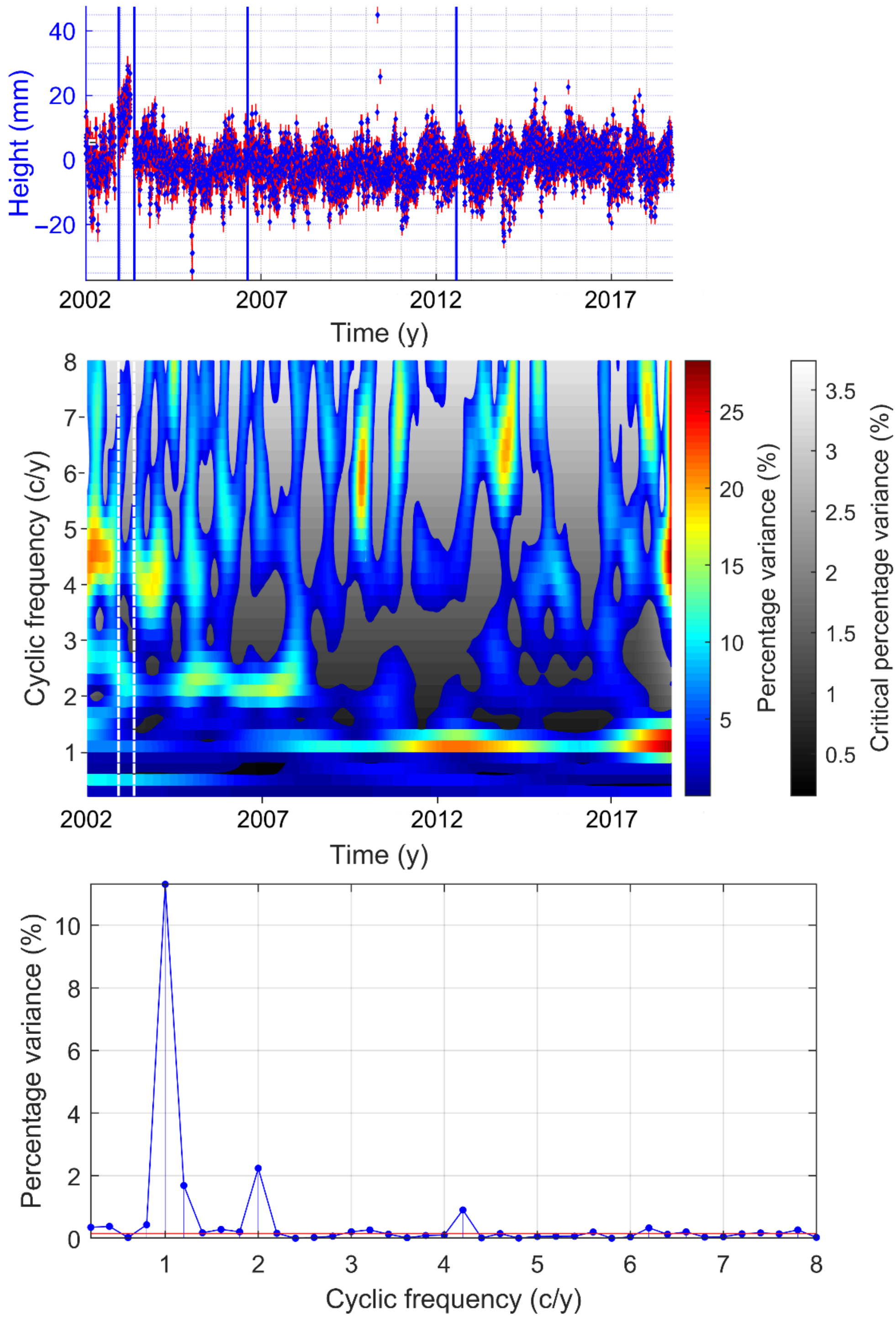

Fig. 3 The GPS height time series for PRDS and its spectral analyses. The top panel shows the series with its error bars in red and datum breaks shown by blue bars. The middle panel is the residual spectrogram with stochastic surface in gray, and the bottom panel is the spectrum with its critical value at 99% confidence level in red

Table 1 The ALLSSA results for the PRDS GPS height time series using the given set of frequencies at 99% confidence level

Freq. (c/y)	Amplitude (mm)	Phase (radians)	Intercepts (mm)	Slope (mm/y)
0.240	1.591 ± 0.000	-0.406 ± 1.365	0.210 ± 0.426	0.096 ± 0.043
0.350	1.367 ± 0.122	-2.134 ± 0.052	12.725 ± 0.603	
1.012	2.787 ± 0.097	2.247 ± 0.035	-1.455 ± 0.219	
1.200	0.897 ± 0.099	1.992 ± 0.111	-3.848 ± 0.357	
2.002	1.217 ± 0.069	-1.204 ± 0.086	-1.787 ± 0.607	
4.216	0.901 ± 0.114	0.774 ± 0.074		

The middle panel in Fig. 3 shows the residual spectrogram with its stochastic surface at 99% confidence level, using the Morlet wavelet. The annual peaks are stronger from year 2010 to year 2016. On the other hand, the semi-annual peaks are stronger from year 2004 to year 2009. A possible explanation could be the impact of weather on the presence of ground water, contributing to a semi-annual behavior of the motions of the geodetic monuments. The measurement errors and noise might partially contribute to this behavior too. The spectrogram shows inter-annual peaks at 0.35 c/y that are more significant from year 2002 to year 2006. This inter-annual behavior may also be linked to the warming effect that could cause monument motion. Furthermore, the short duration bi-monthly peaks at 6 c/y during years 2010 and 2014 are statistically significant. Since the dominant spectral peaks are the annual peaks, one may suppress them to search for any other hidden components.

Discussion

The computational complexity of the LSWA and LSCWA will be dependent on several factors, such as the window size parameters, covariance matrices, number of constituents of known forms, set of frequencies, and the translating windows. The LSWA can be set up to achieve $O(n)$ complexity per frequency like the fast algorithm of the continuous wavelet transform when windows do not overlap, and covariance matrices are ignored. It can also be as slow as

$O(n^3)$ for a series of size n with n frequencies to be examined when the windows overlap. The appropriate selection of the window size parameters can considerably reduce the computational cost. In many fields of science, the quality and reliability of the spectral peaks corresponding to periodic and/or aperiodic signals are more important than the speed. We recommend users to consider splitting series of sizes more than $10{,}000$ for better performance and display purposes using the GUI.

Astronomers are usually interested in analyzing light curves of variable stars whose brightness as seen from the Earth fluctuates. Such astronomical time series are often unequally spaced and weighted. The LSWA is expected to show its robust performance in detecting and quantifying periodic and aperiodic signals in these series. Atmospheric scientists or agronomists may want to interpolate/extrapolate atmospheric temperature time series for various purposes, such as crop disease forecasting (De Wolf et al. 2003). The ALLSSA may be used for these purposes.

Physicians may be interested in studying the coherency between the heart rate variability and brain waves (Niedermeyer and da Silva 2005). The LSCWA may be applied to investigate the coherency between the components of these time series rigorously, or the LSWA may be applied to the brain time series to search for brain tumors and other space-occupying lesions. Financial analysts may be interested in analyzing certain time series to estimate trends in the overall unemployment rate as well as any periodicity in such series, or occasional trends in price movement in intraday transaction prices of International Business Machines (IBM) stock and its coherency with transactions that resulted in price change (Tsay 2010). The ALLSSA and LSWA may be able to accurately estimate such trends simultaneously with other periodic or aperiodic components in such series for making a reliable decision.

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Computer Code

The MATLAB software package (open-access) is available at www.ghader.org, on GitHub at https://github.com/Ghaderpour/LSWAVE-SignalProcessing, and at the GPS Toolbox website at https://www.ngs.noaa.gov/gps-toolbox.

Appendix A. Error estimation of the least-squares coefficients

We show how one may calculate the unbiased covariance matrix of simultaneously estimated coefficients in the LSSA or ALLSSA. Then we show how the GUI calculates the amplitudes and phases of sine waves with their errors.

Let $\mathbf{f} = [f(t)]$ be a time series of size n with associated covariance matrix $\mathbf{C_f}$ and $\mathbf{P} = \mathbf{C_f^{-1}}$. Assume that $\mathbf{f}$ has d significant datum shifts, and so column vectors $\mathbf{\Phi}_1 = [\mathbf{1}_1]$, $\mathbf{\Phi}_2 = [\mathbf{1}_2], \ldots, \mathbf{\Phi}_d = [\mathbf{1}_d]$, of size n whose elements are zeros and ones will estimate the total shifts of data. The elements of each vector are ones if their locations align with a datum shift segment and zeros elsewhere. Assume that $\mathbf{\Phi}_{d+1} = [\mathbf{t}]$, $\mathbf{\Phi}_{d+2} = [\mathbf{t}^2]$, and $\mathbf{\Phi}_{d+3} = [\mathbf{t}^3]$ to estimate a consistent trend for all datum shifts.

Let $\mathbf{\Phi}_{d+4} = \cos(2\pi\omega_1\mathbf{t})$, $\mathbf{\Phi}_{d+5} = \sin(2\pi\omega_1\mathbf{t})$, ..., $\mathbf{\Phi}_{q-1} = \cos(2\pi\omega_k\mathbf{t})$, and $\mathbf{\Phi}_q = \sin(2\pi\omega_k\mathbf{t})$ be the constituents of known forms whose frequencies (ω_k's) are either entered by users in the LSSA or estimated by the ALLSSA. Therefore, $\underline{\mathbf{\Phi}} = [\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_d, \ldots, \mathbf{\Phi}_q]$ is the $n \times q$ matrix of the constituents of known forms. In the LSSA or ALLSSA, the coefficients of constituents of known forms are estimated as follows

$$\underline{\hat{\mathbf{c}}} = \left(\underline{\mathbf{\Phi}}^{\mathrm{T}}\mathbf{P}\,\underline{\mathbf{\Phi}}\right)^{-1}\underline{\mathbf{\Phi}}^{\mathrm{T}}\mathbf{P}\,\mathbf{f} \tag{A.1}$$

that is a column vector of size q. Therefore, the residual series is $\hat{\mathbf{g}} = \mathbf{f} - \underline{\mathbf{\Phi}}\,\underline{\hat{\mathbf{c}}}$. From the covariance law, the covariance matrix of $\underline{\hat{\mathbf{c}}}$ is estimated as

$$\mathbf{C}_{\underline{\hat{\mathbf{c}}}} = \hat{\sigma}_0^2\left(\underline{\mathbf{\Phi}}^{\mathrm{T}}\mathbf{P}\,\underline{\mathbf{\Phi}}\right)^{-1} \tag{A.2}$$

where $\hat{\sigma}_0^2 = (\hat{\mathbf{g}}^{\mathrm{T}}\mathbf{P}\,\hat{\mathbf{g}})/(n-q)$ is unbiased estimator (Wells and Krakiwsky 1971, Chapter 7).

Now from (A.1), suppose that $\hat{c}_1$ and $\hat{c}_2$ are the estimated coefficients of $\cos(2\pi\omega_1\mathbf{t})$ and $\sin(2\pi\omega_1\mathbf{t})$ whose variances $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ and covariance $\hat{\sigma}_{12}$ are obtained from the elements of $\mathbf{C}_{\underline{\hat{\mathbf{c}}}}$ in (A.2), respectively. To find the estimated amplitude $\hat{a}$ and phase $\hat{\theta}$, we use the following equations

$$a\sin(2\pi\omega_1 t + \theta) = a\sin(\theta)\cos(2\pi\omega_1 t) + a\cos(\theta)\sin(2\pi\omega_1 t) \quad \text{(A.3)}$$

$$\cos^2(\theta) + \sin^2(\theta) = 1 \quad \text{(A.4)}$$

Thus, $\hat{c}_1 = \hat{a}\sin(\hat{\theta})$ and $\hat{c}_2 = \hat{a}\cos(\hat{\theta})$, so $\hat{a} = \sqrt{\hat{c}_1^2 + \hat{c}_2^2}$, $\hat{\theta} = 2\tan^{-1}(\hat{a} - \hat{c}_2)/\hat{c}_1$, where $-\pi < \hat{\theta} < \pi$. If $F = F(X, Y)$ is a function of variables X and Y, then the uncertainty or error in F may be obtained after approximation to a first-order Taylor series:

$$\hat{\sigma}_F = \sqrt{\left(\frac{\partial F}{\partial X}\right)^2 \hat{\sigma}_X^2 + \left(\frac{\partial F}{\partial Y}\right)^2 \hat{\sigma}_Y^2 + 2\left(\frac{\partial F}{\partial X}\right)\left(\frac{\partial F}{\partial Y}\right)\hat{\sigma}_{XY}} \quad \text{(A.5)}$$

where $\partial F/\partial X$ is the partial derivative of F with respect to X, $\hat{\sigma}_X^2$ is the variance of X, and $\hat{\sigma}_{XY}$ is the covariance between X and Y (Ku 1966). Using (A.5), we obtain

$$\hat{\sigma}_{\hat{a}} = (1/\hat{a})\sqrt{\hat{c}_1^2\hat{\sigma}_1^2 + \hat{c}_2^2\hat{\sigma}_2^2 + 2\hat{c}_1\hat{c}_2\,\hat{\sigma}_{12}} \quad \text{(A.6)}$$

$$\hat{\sigma}_{\hat{\theta}} = \frac{2\sqrt{((\hat{c}_2/\hat{c}_1)(\hat{a} - \hat{c}_2))^2\,\hat{\sigma}_1^2 + (\hat{a} - \hat{c}_2)^2\hat{\sigma}_2^2 - 2(\hat{c}_2/\hat{c}_1)(\hat{a} - \hat{c}_2)^2\,\hat{\sigma}_{12}}}{|\hat{a}\,\hat{c}_1|(1 + (\hat{a} - \hat{c}_2)^2/\hat{c}_1^2)} \quad \text{(A.7)}$$

that are the errors of $\hat{a}$ and $\hat{\theta}$, respectively.

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