An introduction to the p-adic absolute value

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[1] Partially supported by NSF Grant DMS-1249384.

Question

Can you dissect a square into an odd number of triangles of equal area?

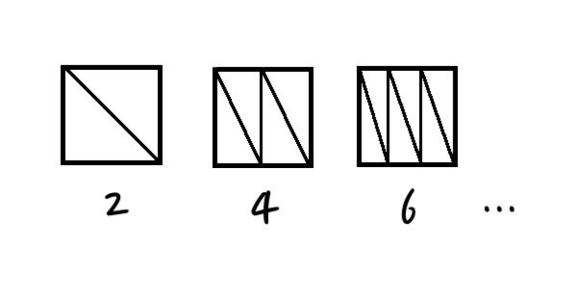

Figure : Equidissection (`https://simpletonsymposium.files.wordpress.com/2013/03/monsky-even-squares.jpg`)

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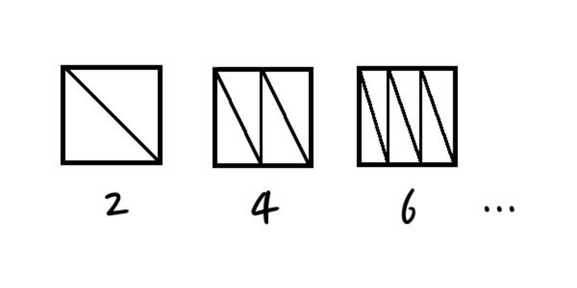

Figure : Equidissection (https://simpletonsymposium.files.wordpress.com/2013/03/monsky-even-squares.jpg)

(This was answered by Paul Monsky in an issue of the MAA's *American Mathematical Monthly* in 1970.)

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Let $n \geq 3$ be an integer. Can you find nonzero integers a, b, and c such that

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(This was answered by Andrew Wiles, who proved Fermat's Last Theorem in the mid-1990s.)

Figure : Andrew Wiles (http://www.simonsingh.net)

Question

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Answer

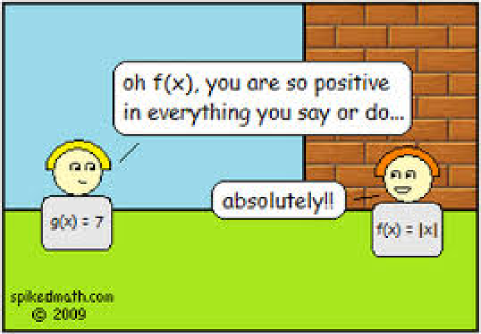
Their answers are both "no," and the proofs both use the p-adic absolute value*.*

Outline

The usual absolute value

$$|x| = \begin{cases} x & x \geq 0 \\ -1 \times x & \text{else} \end{cases}$$

for each real number x.



Key properties of absolute value

Let a and b be real numbers. Then:

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3. $|a + b| \leq |a| + |b|$ (*triangle inequality*)

p-adic absolute value

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with k an integer and c and d integers such that p does not divide cd.

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Example

$p = 2$

$$2 = 2^1 \cdot \frac{1}{1}$$

$$3 = 2^0 \cdot \frac{3}{1}$$

$$3/14 = 2^{-1} \cdot \frac{3}{7}$$

Examples of 2-adic absolute values

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$$|3/14|_2 = |2^{-1} \cdot 3/7|_2 = 2$$

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Absolute values

Remark

For each rational number a,

$$\prod_{p \text{ prime}} |a|_p = |a|^{-1}.$$

p-adic numbers

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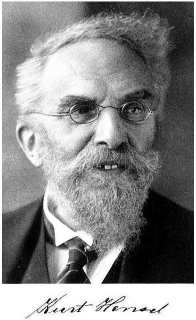
This is the analogue of completing $\mathbb{Q}$ with respect to the usual absolute value to get the ring of real numbers $\mathbb{R}$.

History



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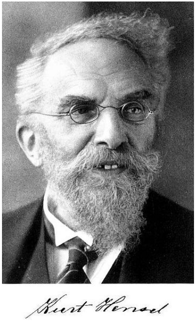

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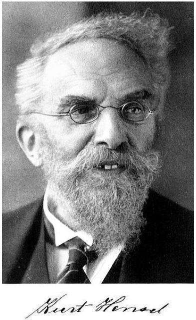

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- Hensel's great uncle was the famous composer Felix Mendelssohn.
- Hensel was an editor of Crelle's Journal.

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Remark

Completely different from the usual absolute value, e.g. $|5+5| = 10$, while $|5| = 5$.

Delightful consequence for convergence of series

As a consequence of the strong triangle inequality, a series

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In other words, you don't need any complicated tests (root test, ratio test, etc) to check convergence of series in the p-adic world!

Congruences

Definition

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So the p-adic absolute value measures congruences.

Convergent sequences

A sequence $a_1, a_2, \ldots$ converges to a p-adically if $|a_i - a|_p \to 0$ as $i \to \infty$, i.e. if $a_i \equiv a$ modulo higher and higher powers of p as $i \to \infty$.

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$1, 1+2, 1+2+2^2, 1+2+2^2+2^3, \ldots$ converges 2-adically. It converges to $1+2+2^2+2^3+2^4+\cdots = \sum_{j=0}^{n} 2^j = \frac{1}{1-2} = -1$.

More convergent sequences

Chose rational numbers b and c. Find a sequence $a_1, a_2, \ldots$ of rational numbers that converges to b with respect to the usual absolute value and converges to c with respect to the p-adic absolute value.

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$$a_n = b\frac{p^n}{p^n - 1} + c\frac{p^n + 1}{p^{2n} + 1}.$$

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Then $a_1, a_2, \ldots$ *converges to* b *with respect to the usual absolute value and converges to* c *with respect to the* p*-adic absolute value.*

Expression of p-adic numbers

- Any nonzero p-adic number a can be expressed uniquely in the form

$$a = a_N p^N + a_{N+1} p^{N+1} + \cdots$$

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- If a is an integer, then this is the base p representation of a.
- Observe that $|a|_p = p^{-N}$.

The trival absolute value

Definition

The trivial absolute value is defined by

$$|x| = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

for each number x.

Ostrowski's Theorem

Question

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Answer (Ostrowski's Theorem, 1916)

Every nontrivial absolute value is equivalent to either the usual absolute value or to a p-adic absolute value. (The absolute values equivalent *to an absolute value* $|\cdot|_*$ *are the absolute values of the form* $|\cdot|_*^c$ *with* $c > 0$*.)*

Generalizations

Can extend the p-adic absolute value to other fields, and can build an analogue of $\mathbb{C}$ denoted $\mathbb{C}_p$

Extensions

Remark

$|\cdot|_p$ can be extended from $\mathbb{Q}$ to $\mathbb{R}$.

We'll need this fact later in the talk.

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Observation

Note the difference from the situation in $\mathbb{Q}$. After completing $\mathbb{Q}$ with respect to the usual absolute value, we obtain $\mathbb{R}$, which does not contain a square root of -1.

Square roots of -1

We're sorry. The number you have dialed is purely imaginary. Please rotate your phone 90 degrees, and dial again.

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4. p-adic geometry and p-adic analysis

Some surprising geometric consequences

Recall the strong triangle inequality: For any p-adic numbers a and b

$$|a+b|_p \leq \max\left(|a|_p, |b|_p\right)$$

with equality if $|a|_p \neq |b|_p$.

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- If two intervals intersect, then one contains the other. (Can extend this result to discs, etc.)
- All triangles are isosceles.
- The only *connected* subsets are the one point sets, i.e. $\mathbb{Q}_p$ is *totally disconnected*.

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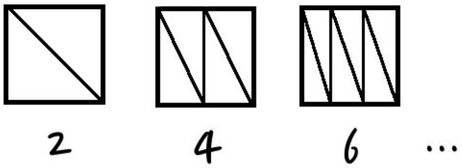

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Dissecting a square into triangles of equal area

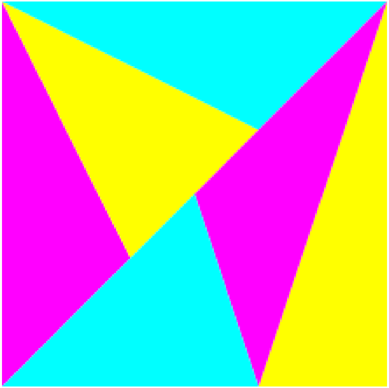

Figure : Equidissection (wikipedia)

Problem history

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- Richman, who had initially wanted to put the problem on an exam, had shown impossible for $n = 3$ and for $n = 5$.

Problem history

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- Richman also proved that if a square can be dissected into n triangles of equal area, then the same is true for $n + 2$.
- Monsky answered it (building on John Thomas's work) in 1970, in the MAA's American Mathematical Monthly.

Monsky's Theorem

Theorem (Monsky, 1970)

It is impossible to dissect a square into an odd number of triangles of equal area.

Figure : Paul Monsky (wikipedia)


I HATE BEING THE ODD ONE OUT...
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Outline of proof of Monsky's Theorem

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- Note that if n is odd, then $\left|\frac{1}{n}\right|_2 = 1$, and if n is even $\left|\frac{1}{n}\right|_2 > 1$.
- We'll show that if unit square can be dissected into n triangles of equal area, then $\left|\frac{1}{n}\right|_2 > 1$, i.e. n is even.

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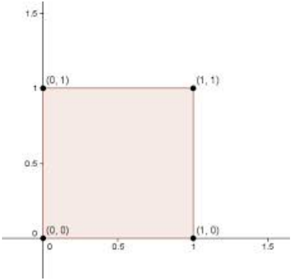

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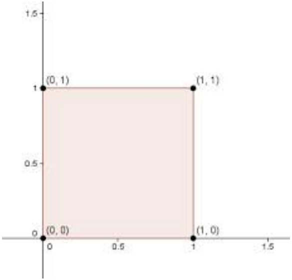

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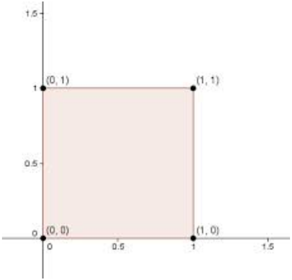

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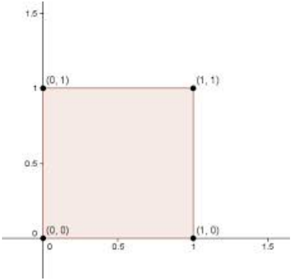

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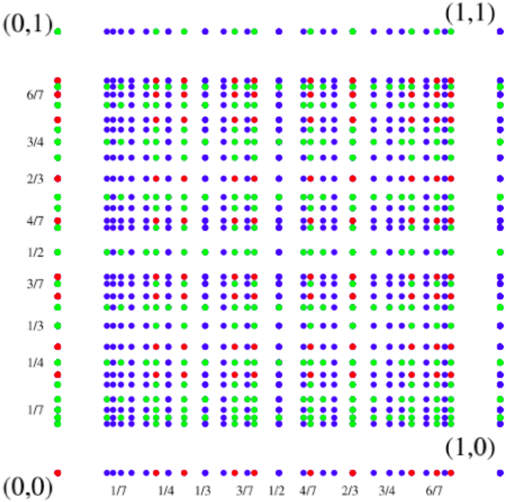

Figure : Coloring of the unit square (from *Proofs From the Book*)

Outline of proof of Monsky's Theorem

Step 3: Show that any blue point $P_b = (x_b, y_b)$, red point $P_r = (x_r, y_r)$, and green point $P_g = (x_g, y_g)$ form the vertices of a triangle of positive area A, and $|A|_2 > 1$.

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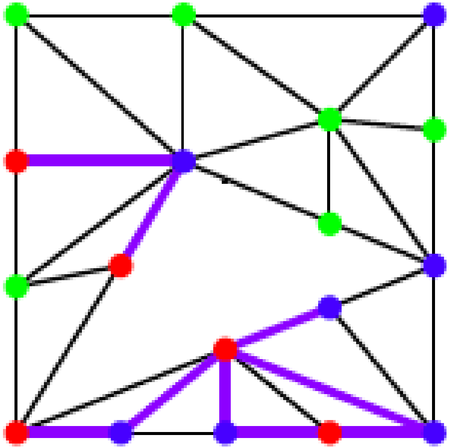
   So $|A|_2 > 1$.

A dissection of the square



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Step 4: Show that every dissection of the unit square into a finite set of triangles contains at least one triangle whose vertices are three different colors (i.e. a triangle whose area A satisfies $|A|_2 > 1$).

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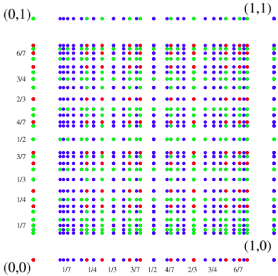

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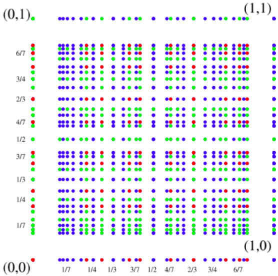

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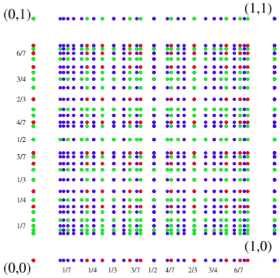

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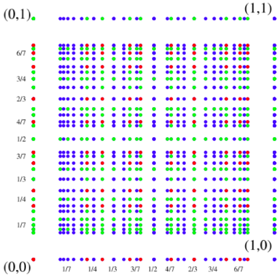

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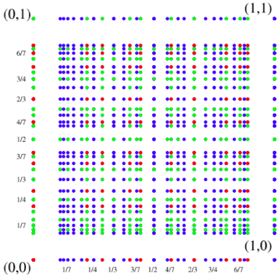

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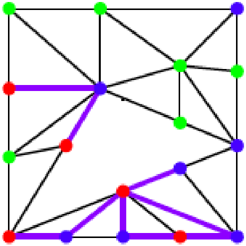
3. Consequence: At least one triangle has an odd number of red-blue segments, which in turn implies it has vertices of 3 different colors.

Summary



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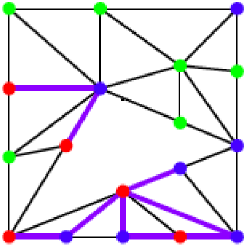

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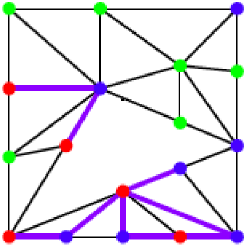

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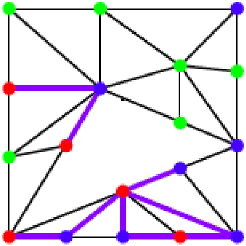

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- Using determinants, we showed that the area of such a triangle has 2-adic absolute value > 1.
- In other words, the area of such a triangle cannot be $1/n$ with n odd.

Conclusion

It is impossible to dissect a square into an odd number of triangles of equal area.

Generalizations of Monsky's Theorem

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What about dissections of other polygons into triangles of equal area?

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3. Equidissection of general quadrilaterals studied by Du, Ren, and Su

Happy π Day Eve!

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