Contradiction

In proof by contradiction, the proof starts by assuming the conclusion is false and the prove proceed to show that this is impossible by contradicting some fact. This fact could be some definition of mathematics or a prerequisite given in the problem. Consider the following three examples. The first two are famous results in mathematics. The third example is a problem taken from a mathematics competition.

Example 1
Prove that $\sqrt{2}$ is irrational. A number is irrational if it cannot be written as a fraction where the numerator and denominator are integers.

Solution
Assume that $\sqrt{2}$ is rational. Then we can write $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$. We can rearrange $\sqrt{2} = \frac{a}{b}$ into $2b^2 = a^2$. Since 2 divides $2b^2$ then 2 must also divide a^2. Thus, 2 must divide a. Therefore, we can write $a = 2k$ where $k \in \mathbb{Z}$. Substituting this into $2b^2 = a^2$ gives $b^2 = 2k^2$. Using similar logic, 2 must divide b. This is a contradiction to $\gcd(a, b) = 1$. Therefore, $\sqrt{2}$ is irrational.

Example 2
Prove that there are infinite number of primes.

Solution
Assume that there are finite number of primes, $p_1, p_2, \dots, p_n$. Now let $n = p_1 p_2 \cdots p_n + 1$. Since n is obvious larger than any p_i, n cannot be a prime. Therefore, there must exist a p_i that divides n. Since p_i must divide $p_1 p_2 \cdots p_n$, then p_i must divide 1. However, there does not exist a prime that divide 1. Thus, we have a contradiction.

Example 3
Prove that there is no polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

with integer coefficients and of degree at least 1 with the property that $P(0), P(1), \dots$ are all prime numbers.

Solution
Assume that $P(0), P(1), \dots$ are all prime numbers. In particular, $P(0) = a_0$ is a prime number. Next, observe that for every integer $k \geq 1$,

$$P(ka_0) = a_n (ka_0)^n + a_{n-1} (ka_0)^{n-1} + \cdots + a_1 (ka_0) + a_0$$

Thus, $P(a_0), P(2a_0), P(3a_0), \dots$ are all divisible by a_0 and is prime. Therefore,

$$a_0 = P(a_0) = P(2a_0) = P(3a_0) = \dots$$

Therefore, the polynomial $P(x) - a_0$ has

$$a_0, 2a_0, 3a_0, \dots$$

as roots. However, a polynomial cannot have infinite roots. This is a contradiction. Therefore, the original is true.

Practice Problems

1. Prove that $\sqrt{3}$ is irrational.
2. Prove that the sum of a rational number and irrational number is irrational.
3. Prove that there does not exist a rational number r such that $2^r = 3$.
4. Let a, b be real numbers such that $ab = 0$. Prove that either $a = 0$ or $b = 0$.
5. Let a, b, c be integers such that $a^2 + b^2 = c^2$. Prove that abc must be even.
6. The product of 34 integers is equal to 1. Prove that their sum cannot be 0.
7. Let n be a natural number that is not a perfect kth power. Prove that $\sqrt[k]{n}$ is irrational.
8. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.
9. Prove that there does not exist a rational number r such that $10^r = 15$.
10. Prove that at any party there are two people who have the same number of friends at the party (assume that all friendships are mutual).
11. Prove that the sum of two odd perfect squares cannot be a perfect square.
12. Prove that for all integer a, b, c such that $a^2 + b^2 = c^2$ then either a is even or b is even.
13. In a ping pong tournament, each player plays every other player exactly once. Prove that there is some player in the unfortunate position that every other player either beat him or beat someone who did.
14. Given that a, b, c are odd integers, prove that the equation $ax^2 + bx + c = 0$ cannot have a rational roots.
15. If p_n is the nth prime number, prove that

    $$p_1 p_2 \cdots p_n + 1$$

    is never a perfect square.
16. Prove that there are infinite number of primes of the form $4k + 3$ where k is a positive integer.
17. Let $a_1, a_2, \ldots, a_n$ be positive integers such that

    $$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1$$

    Prove that whenever n is an even number, at least one of the a_i is even.
18. Let $p(x)$ be a polynomial with integer coefficients such that $p(0)$ and $p(1)$ are odd. Prove that p has no integer zeros.
19. Show that there does not exist a function $f : \mathbb{Z} \to \{1, 2, 3\}$ satisfying $f(x) = f(y)$ for all $x, y \in \mathbb{Z}$ such that $|x - y| \in \{2, 3, 5\}$
20. Determine all real solutions to the system of equations

    $$\begin{aligned} x + \log_{10} x &= y - 1 \\ y + \log_{10}(y - 1) &= z - 1 \\ z + \log_{10}(z - 2) &= x + 2 \end{aligned}$$