VANDERBILT UNIVERSITY School of Engineering

Discrete Structures

CS 2212

(Fall 2020)

26 – Counting

Reminder and Recap …

Recap:

- **Last time**
  - We started Chapter 9 (Counting)
  - Permutations and Combinations (*without* repetitions)
- **Today**
  - Permutations and Combinations (*with* repetitions)
  - Pigeonhole Principle

Counting – Permutations and Combinations

Consider a set (collection) of ***n*** objects.

Set = { α , β , γ , . . . , δ }

What is the total number of **Sequences** of length ***r***?

[____ ____ ____ ____ ____]

- Order is important
- Permutation problem
- Elements in the sequence are not repeated (so far).

$$\boxed{P(n,k) = \frac{n!}{(n-r)!}}$$

Counting – Permutations and Combinations

Consider a set (collection) of $\boldsymbol{n}$ objects.

$$\text{Set} = \{\, \alpha\, ,\, \beta\, ,\, \gamma\, , \ldots ,\, \delta\, \}$$

What is the total number of **Subsets** of length $\boldsymbol{r}$?

{____ ____ ____ ____ ____}

- Order is not important
- Combination problem
- Elements in the subset are not repeated (so far).

$$\boxed{\binom{n}{r} = \frac{n!}{r!\,(n-r)!}}$$

Counting with Repetitions

So far, our setup only considered selections in which **items cannot be repeated**.

For instance: Set = $\{ \alpha, \beta, \gamma, \delta \}$

Select a sequence of length three (*with no repetitions*.)
e.g., $[\alpha, \beta, \gamma]$

or

Select a subset of length three (*with no repetitions*.)
e.g., $\{\alpha, \beta, \gamma\}$

How to count the possible number of selections if items can be repeated?

For instance, $[\alpha, \gamma, \alpha]$ is a valid sequence selection, or $\{\alpha, \gamma, \alpha\}$ is a valid subset selection.

Permutation with Repetitions

Multiset: A set in which elements may be repeated.

Example: $S = \{A, A, B, B, B\}$

Also referred to as the **bag** in the context of counting.

Problem: Assume our collection is a multiset with a total of n elements, of which k are distinct. How many permutations are there?

In other words,

how many *distinct sequences* of length n can be obtained?

Permutation with Repetitions

Find the number of possible permutations of the letters in the word babbage.

Here, n = 7 (total spots)

a	b	a	b	g	b	e

7 possible locations for three b's

4 possible locations for two a's

2 possible locations for one g

1 possible locations for one e

$$\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1} = \boxed{\frac{7!}{3!2!1!1!}}$$

Permutation with Repetitions

The number of permutations of an n-element multi-set with k distinct elements, where the i^{th} distinct element is repeated n_i times is

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Permutation with Repetitions

Question:

Find the number of possible permutations of the letters in the word babbage.

Answer:

$$\frac{7!}{3!2!} = 420$$

Permutation with Repetitions

A camp offers 3 different activities: hiking, swimming, and crafts. The capacity in each activity is limited so that 5 kids can do hiking, 4 can do crafts and 6 can do swimming. There are 15 kids in the camp. How many ways are there to assign the kids to the activities?

activities	H	H	H	H	H	C	C	C	C	S	S	S	S	S	S
kids	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

$$\binom{15}{5}\binom{10}{4}\binom{6}{6} = \frac{15!}{5!4!6!}$$

Permutation with Repetitions

Example: S = { A, A, B, B, B}

There are 5! permutations of this set. **BUT…**

We don't want to count permutations that are identical and repeated. After all blue A and red A are both the same, that is A.

So, in our 5! permutations,

____ __A__ _____ __A__ _____ is different than

____ __A__ _____ __A__ _____

and both are counted. But, actually, they are same. So, we need to remove these duplications.

Permutation with Repetitions

Thus, we need to divide 5! by the number of permutations of each identical element: 2! for A and 3! for B.

That is, the unique permutations will be

$$\frac{5!}{2!3!} = \frac{5 \times 4 \times 3!}{2!3!} = 10 \text{ permutations}$$

B B B A A	B A A B B
B B A B A	A B B B A
B B A A B	A B B A B
B A B B A	A B A B B
B A B A B	A A B B B

Counting Multisets

Recall multisets – A set that may contain repeated elements. (Order is not important).

Example: $S = \{ A, B, A, C\}$

Let there be k different types of objects.

- All objects of the same type are indistinguishable.
- There is no limitation on number of objects of each type.

Under these conditions, what is the number of ways to select n objects from a set of k varieties ?

Counting Multisets

Example:

Lets go to a shop with four different types of chocolates.

A customer wants to fill a box with 10 chocolate bars.

$k = 4$

$n = 10$

A selection of 10 chocolate bars is a multiset.

How many such selections are possible if four varieties of chocolates are available?

Counting Multisets

If there is no limitation on the number of each variety available and objects of the same variety are indistinguishable, the number of ways to select $\boldsymbol{n}$ objects from a set of $\boldsymbol{k}$ varieties is:

$$\binom{n+k-1}{k-1}$$

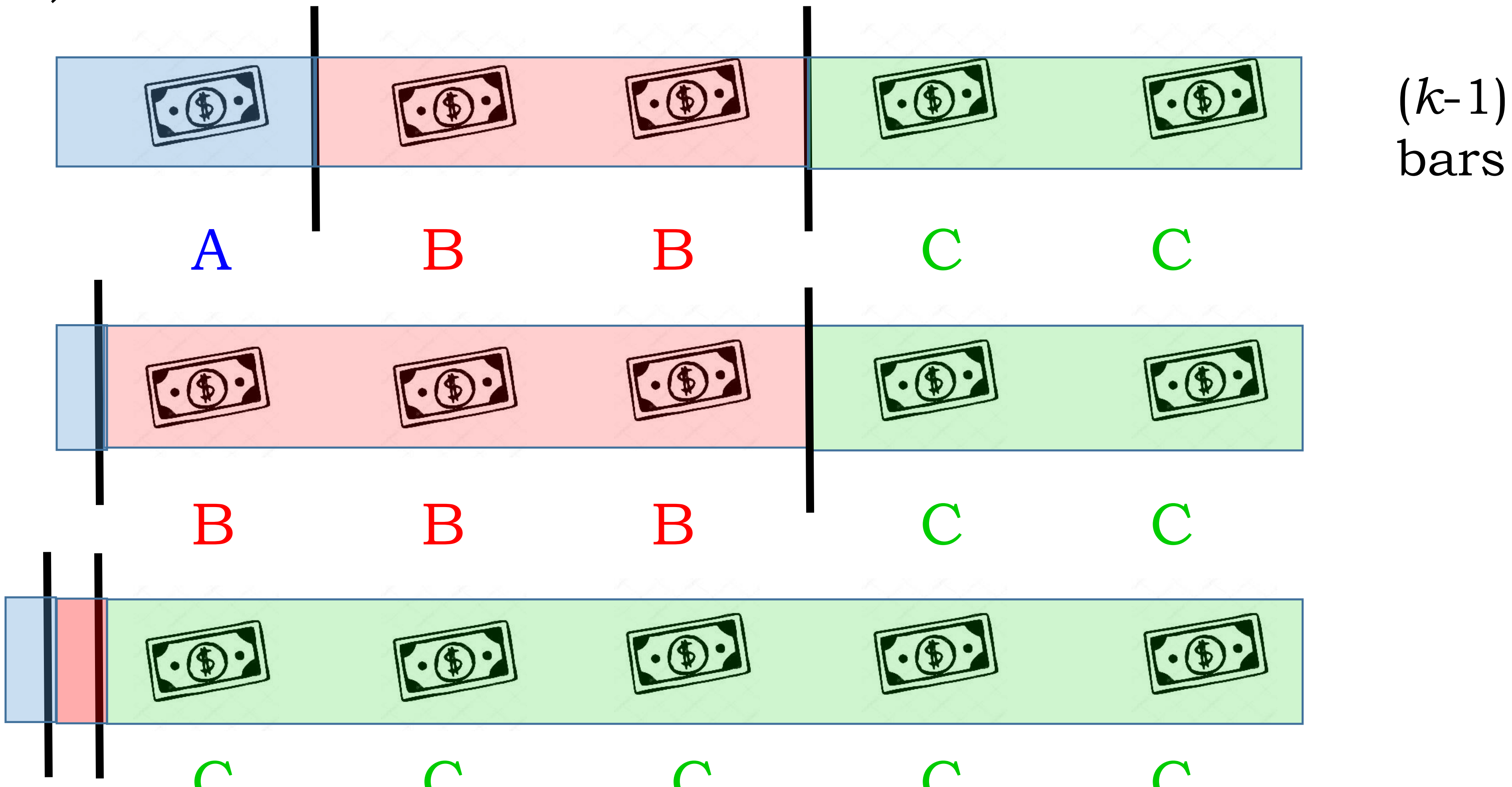
How do we get this formula?

Counting Multisets

There are five 1$ bills. In how many ways can we distribute them to A, B, and C?

$n = 5$

$k = 3$



Counting Multisets

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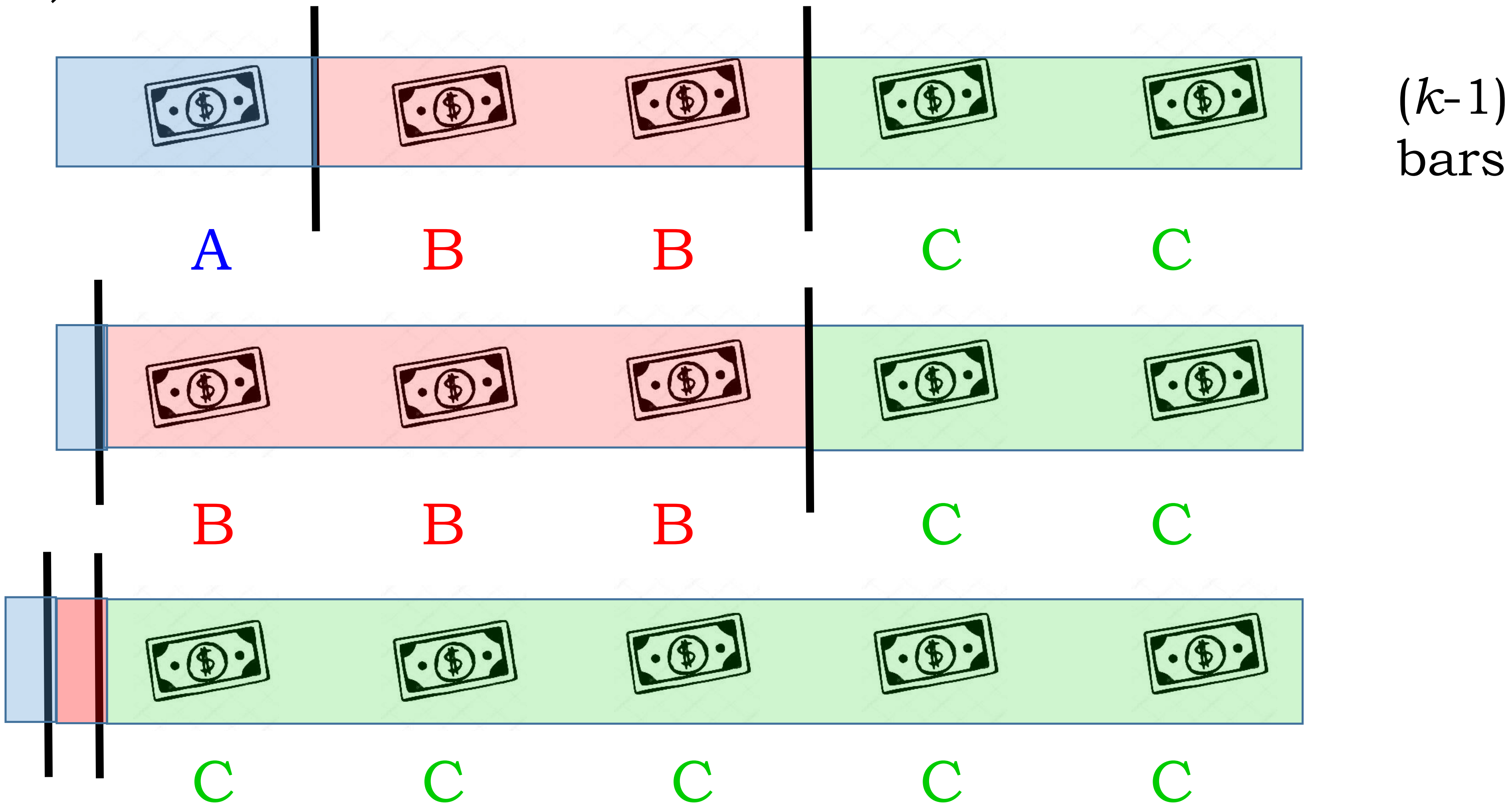


Counting Multisets

In general, it’s a star and bar problem

$$* \quad * \mid * \mid * \quad * \quad * \mid *$$

n stars
(k-1) bars

Number of ways to select n objects from a variety of k objects is

$$\binom{stars + bars}{bars} = \binom{n + k - 1}{k - 1}$$

Counting Multisets

Example:

A customer wants to fill a box with 10 chocolate bars. How many such selections are possible if four varieties of chocolates are available?

$n = 10$

$k = 4$

$$\binom{n+k-1}{k-1} = \binom{13}{3} = 286$$

Counting Multisets

Example:

A large number of coins are divided into four piles according to whether they are pennies, nickels, dimes or quarters. How many ways are there to select 25 coins from the piles?

n = 25 (size of the multiset to be selected)

k = 4 (types of coins)

$$\binom{n+k-1}{k-1} = \binom{28}{3}$$

Counting Multisets

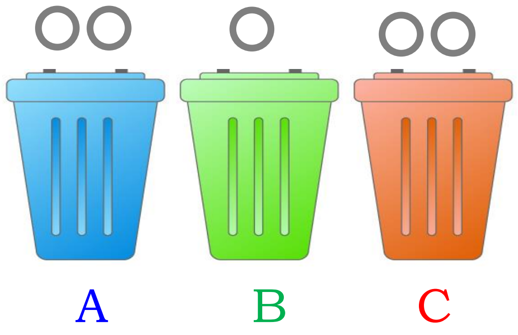
Another way to think about this problem is in terms of **"Bins and Balls"** problem.

n	Size of multiset to be selected	Number of balls
k	Number of varieties	Number of bins

Example: 5 balls and 3 bins.

Balls are indistinguishable.

Bins are distinguishable.



The number of ways to place n indistinguishable balls into m distinguishable bins is:

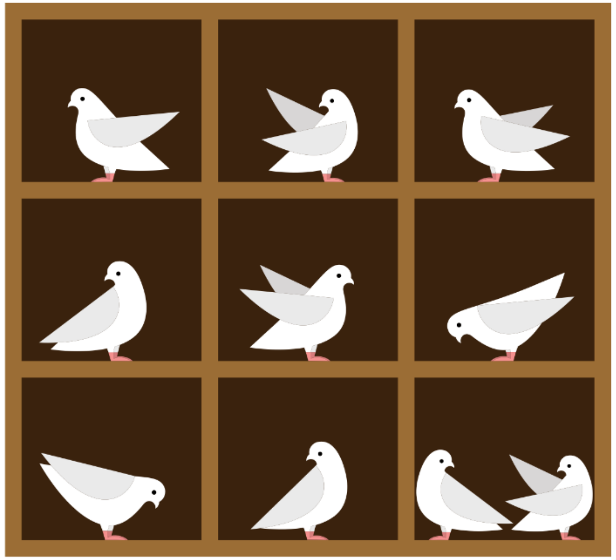
$$\binom{bins + balls - 1}{balls - 1}$$

The Pigeonhole Principle

The Pigeonhole Principle

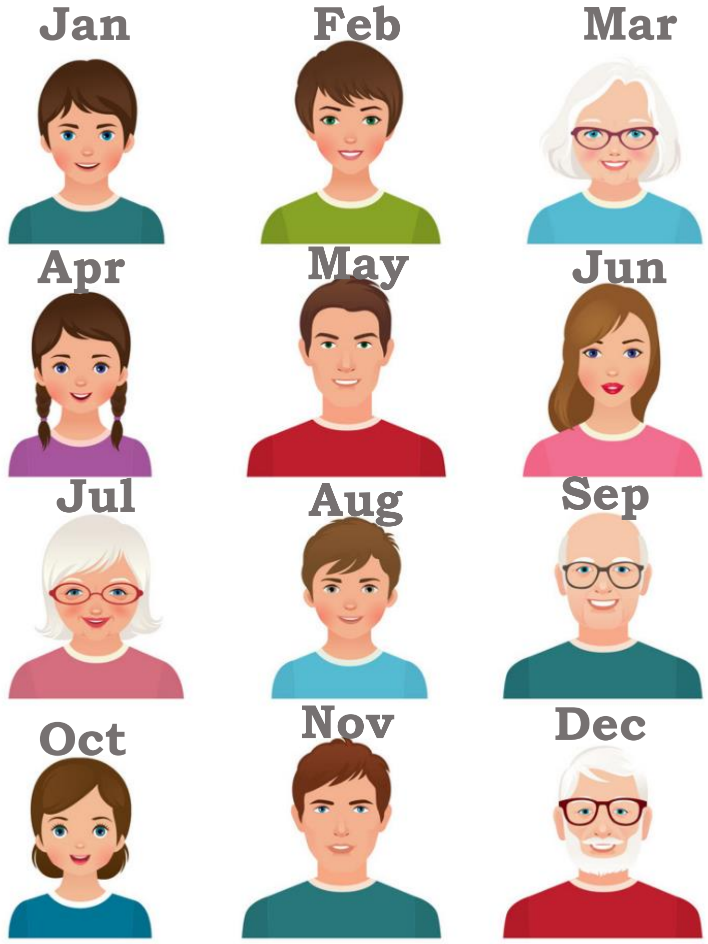
If A and B are finite sets with $|A| > |B|$, then there are no injective functions from A to B. This is called the **pigeonhole principle**.

If m things are to be put into n places and m > n, then one place has two or more things in it.



The Pigeonhole Principle – Examples

What is the smallest number of people in a group, so that it is guaranteed that at least two of them will have their birthday in the same month?



Answer: 13

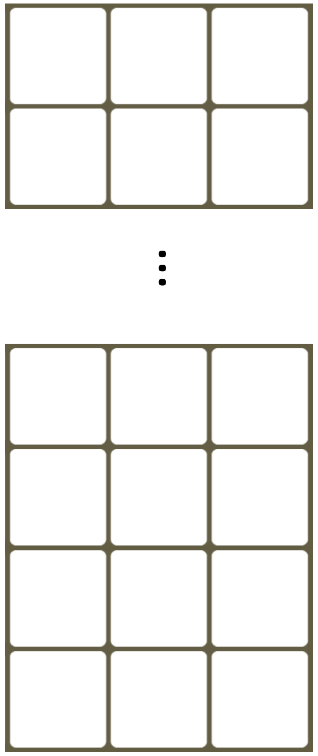
Pigeonhole principle

In general,

If $n = km + 1$ objects are distributed among m sets, then at least one of the sets will contain at least $(k + 1)$ objects.

The Pigeonhole Principle – Examples

Consider a grid of (4×82) points such that each point is colored either red, green, or blue. Prove that there always exists a rectangle in the grid such that all four of its vertices are the same color.



The Pigeonhole Principle – Examples

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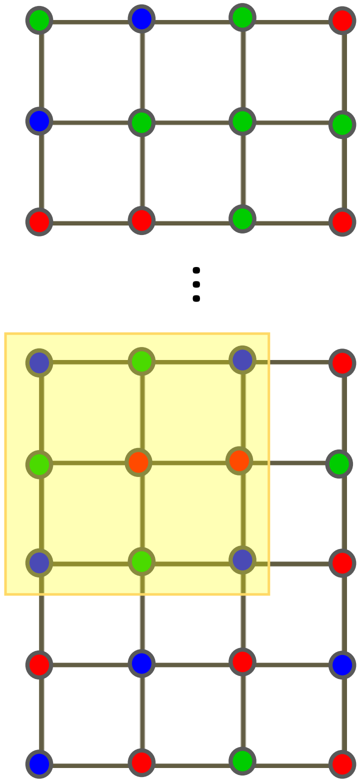
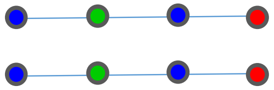


- Each row has 4 points that need to be colored using 3 colors.
- If there are two rows whose points are colored the same way, we know the required rectangle exists.



- So, all we need to show is there always exist such two rows.
- Each row can be colored in one of the $3^4 = 81$ ways.
- So, if we have more than 81 rows, two rows will always exist whose points have same colors. Why?
- By the Pigeonhole principle. Hence, the desired result follows.

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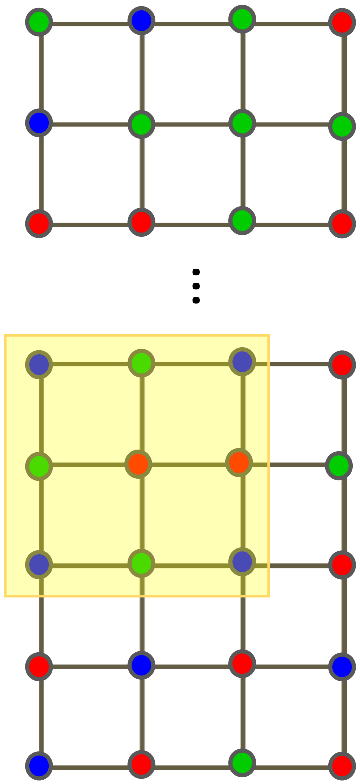


Pigeon holes:	Coloring options
Pigeons:	Number of rows

The Pigeonhole Principle – Examples

In a group of 680 students, is it true that there are always two that have the same first and last initial?

Yes!

- There are 26 x 26 = 676 possible combinations of initials.
- So in the worst case, you would go through 676 people without finding a matched set of initials.
- But you will find a match with the 677^{th} person's initials.

Pigeon holes:	Letter combinations for initials
Pigeons:	Students (680)

The Generalized Pigeonhole Principle

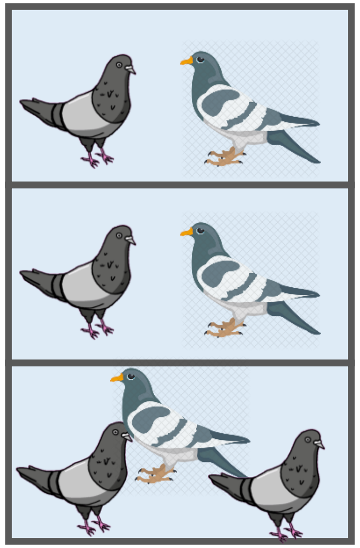
Consider a function whose domain has n elements and whose target has k elements, for n and k positive integers. Then there is an element y in the target such that f maps at least $\lceil n/k \rceil$ elements in the domain to y.

Example:

$n = 7$

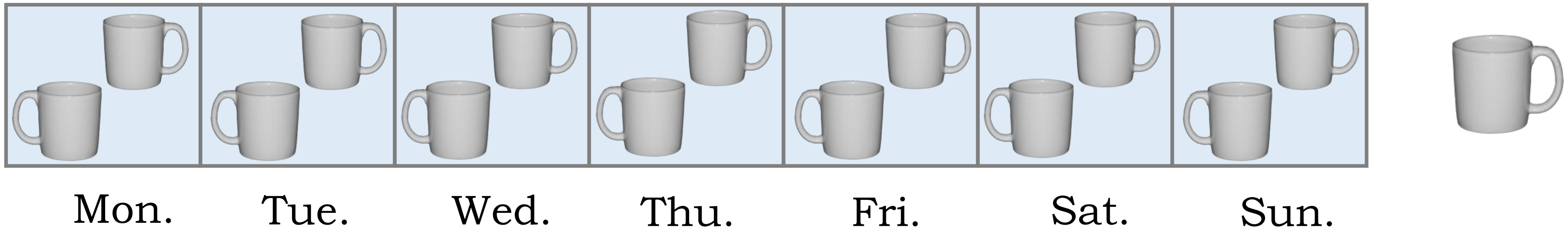
$k = 2$

$\lceil n/k \rceil = 3$



The Generalized Pigeonhole Principle

Alice bought 15 cups of coffee in a week. Then there was a day in the week in which she bought at least 3 cups of coffee.



(Total coffee cups)	n	$=$	15	(pigeons)
(No. of weekdays)	k	$=$	7	(pigeonholes)

$$\lceil n/k \rceil = 3$$

The Generalized Pigeonhole Principle

There are 121.4 million people in the United States who earn an annual income that is at least $10,000 and less than $1,000,000. Annual income is rounded to the nearest dollar. Show that there are 123 people who earn the same annual income in dollars.

(Total population) $n = 121{,}400{,}000$ (pigeons)

(No. of possible incomes) $k = 990{,}000$ (pigeonholes)

$\lceil n/k \rceil = 123$