Signed Group Orthogonal Designs And Their Applications

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Orthogonal designs

Definition

An *orthogonal design*, OD, of order n and type $(u_1, \ldots, u_\ell)$, denoted $OD(n;\ u_1, \ldots, u_\ell)$, is a square matrix X of order n with entries from $\{0, \pm x_1, \ldots, \pm x_\ell\}$, where the x_j's are commuting variables, that satisfies

$$XX^t = \left(\sum_{j=1}^{\ell} u_j x_j^2 \right) I_n,$$

where X^t denotes the transpose of X, and I_n is the identity matrix of order n.

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- An OD in which there is no zero entry is called a full OD.

Example

- Let $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. It can be seen that $AA^t = (a^2 + b^2)I_2$, and so A is an $OD(2; 1, 1)$.

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- Let $B = \begin{bmatrix} a & b & c & c \\ -b & a & c & -c \\ c & c & -a & -b \\ c & -c & b & -a \end{bmatrix}$. It can be seen that

$$BB^t = (a^2 + b^2 + 2c^2)I_4,$$

and so B is an $OD(4; 1, 1, 2)$.

Hadamard matrices

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- Equating all variables of any full OD to 1 results in a Hadamard matrix.
  It is conjectured that a Hadamard matrix of order $4n$ exists for each $n \geq 1$.
- J. Seberry obtained the first asymptotic existence result for Hadamard matrices, namely, for any odd integer $q > 3$, there is a Hadamard matrix of order $2^n q$ for every

$$n \geq 2\log_2(q-3).$$

Radon's number

- The number of variables in an OD of order $n = 2^a b$, b odd, cannot exceed $\rho(n)$ (Radon's number), where

$$\rho(n) = 8c + 2^d,$$

and c, d are obtained from $a = 4c + d$, $0 \leq d < 4$.

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Example

The maximum number of variables in ODs of orders $12 = 2^2 \cdot 3$ and $32 = 2^5 \cdot 1$ are 4 and 10, respectively.

The asymptotic existence of orthogonal designs

Theorem (P. Eades, P. Robinson and J. Seberry)

Suppose that there is an $OD(d;\ w_1, \dots, w_m)$, *where* $w_1, w_2, \dots, w_m$ *are powers of 2 and* $w_1 + \cdots + w_m = d$.

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Suppose that there is an $OD(d;\ w_1, \dots, w_m)$, where $w_1, w_2, \dots, w_m$ are powers of 2 and $w_1 + \cdots + w_m = d$.
Then for every k-tuple $(u_1, \dots, u_k)$ of positive integers such that $u_1 + \cdots + u_k = 2^a d$, there is an integer $N = N(u_1, \dots, u_k)$ such that for each $n \geq N$, there is an

$$OD(2^{n+a} d;\ 2^n u_1, \dots, 2^n u_k).$$

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Theorem (E. Ghaderpour and H. Kharaghani)

For any k-tuple $(u_1, \dots, u_k)$ of positive integers, there is an integer $N = N(u_1, \dots, u_k)$ such that a full OD of type

$$(2^n u_1, \dots, 2^n u_k)$$

exists for each $n \geq N$.

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- The complex signed group $S_{\mathbb{C}} = \langle i;\ i^2 = -1 \rangle = \{\pm 1, \pm i\}$ is a group of order four. This is the smallest non-trivial signed group.
- The set of all monomial $\{0, \pm 1\}$-matrices of order n, denoted SP_n, forms a group of order $2^n n!$.

Signed group orthogonal designs

Definition

A *signed group orthogonal design*, SOD, of type $(u_1, \ldots, u_k)$, where $u_1, \ldots, u_k$ are positive integers, and of order n, is a square matrix X of order n with entries from $\{0, \epsilon_1 x_1, \ldots, \epsilon_k x_k\}$, where the x_j's are commuting variables and $\epsilon_j \in S$, $1 \leq j \leq k$, for some signed group S, that satisfies

$$XX^* = \left(\sum_{i=1}^{k} u_i x_i^2\right) I_n,$$

where X^* is the transpose conjugate of X. We denote this SOD by $SOD(n;\ u_1, \ldots, u_k)$.

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- An SOD over the complex signed group $S_{\mathbb{C}}$ results in a complex OD, COD.
- An SOD over the trivial signed group $S_{\mathbb{R}}$ results in an OD.

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A *real monomial representation* (*remrep*) of degree n is a signed group homomorphism $\pi : S \to SP_n$:

- $\pi(ab) = \pi(a)\pi(b) \quad \text{for all} \quad a, b \in S$,
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- An SOD with no zero entries is called a full SOD. Equating all variables to 1 in any full SOD results in a signed group Hadamard matrix, SH.

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Theorem (R. Craigen)

For any odd positive integer q, there exists a circulant SH of order $2q$ over a signed group S that admits a remrep of degree $2^{2N(q)-1}$.

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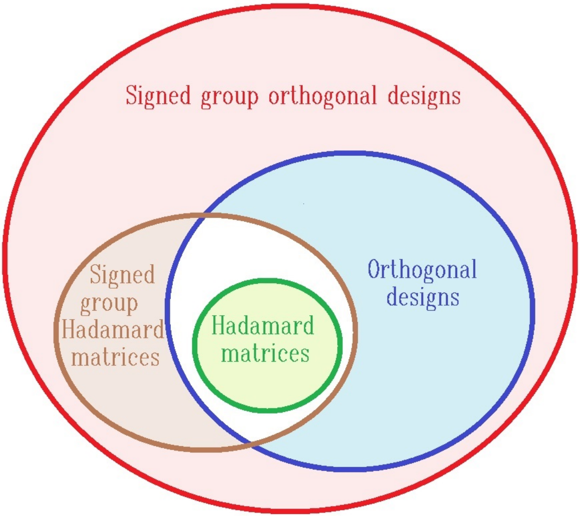
- R. Craigen and I. Livinskyi showed that for any odd integer $q > 1$, there exists a Hadamard matrix of order $2^n q$ for every

$$n \geq \frac{1}{5}\log_2\left(\frac{q-1}{2}\right) + 13.$$


Signed group orthogonal designs
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An application of signed group orthogonal designs

Theorem

Suppose that there exists a $SOD(n;\ u_1, \ldots, u_k)$ for some signed group S equipped with a remrep π of degree m, where m is the order of a Hadamard matrix. Then there exists an

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Example

A $COD(n;\ u_1, \ldots, u_k)$ can be viewed as a $SOD(n;\ u_1, \ldots, u_k)$ over the complex signed group $S_{\mathbb{C}}$. Define $\pi : S_{\mathbb{C}} \to SP_2$ by

$$i \longrightarrow \begin{bmatrix} 0 & 1 \\ - & 0 \end{bmatrix}.$$

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It can be verified that π is a remrep of degree 2, and so by the previous theorem, there exists an $OD(2n;\ 2u_1, \ldots, 2u_k)$.

Definition

Let S be a signed group, and $A = [a_{ij}]$ be a square matrix such that $a_{ij} \in \{0, \epsilon_1 x_1, \ldots, \epsilon_k x_k\}$, where $\epsilon_\ell \in S$ and x_ℓ is a variable, $1 \leq \ell \leq k$.

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For each $a_{ij} = \epsilon_\ell x_\ell$ or 0, let $\overline{a}_{ij} = \overline{\epsilon}_\ell x_\ell$ or 0, and $|a_{ij}| = |\epsilon_\ell x_\ell| = x_\ell$ or 0. We define $abs(A) := [|a_{ij}|]$.

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We call A is *quasisymmetric*, if

$$abs(A) = abs(A^*),$$

where $A^* = [\overline{a}_{ji}]$.

Example

The following matrix is a circulant matrix of order 3 over the complex signed group $S_{\mathbb{C}}$:

$$C = \begin{bmatrix} ia & b & -ib \\ -ib & ia & b \\ b & -ib & ia \end{bmatrix}.$$

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So, $C = \mathrm{circ}\big(ia, b, -ib\big)$ and $C^* = \mathrm{circ}\big(-ia, ib, b\big)$.
The circulant matrix C is also quasisymmetric because

$$abs(C) = abs(C^*) = \mathrm{circ}\big(a, b, b\big).$$

Asymptotic existence of signed group orthogonal designs

Theorem

Suppose that $(u_1, u_2, \ldots, u_k)$ *is a k-tuple of positive integers and let* $u_1 + \cdots + u_k = u$. *Then there is a full circulant quasisymmetric*

$$SOD(4u;\ 4u_1, 4u_2, \ldots, 4u_k)$$

for some signed group S *that admits a remrep of degree* 2^n, *where depending on the sequences used to create this SOD,*

$$n \leq \frac{3}{13} \sum_{i=1}^{k} \log(u_i) + 8k + 2,$$

or

$$n \leq \frac{1}{5} \sum_{i=1}^{k} \log(u_i) + 10k + 2.$$

An application of signed group orthogonal designs

Theorem

Suppose that $(u_1, u_2, \ldots, u_k)$ *is a* k*-tuple of positive integers and let* $u_1 + \cdots + u_k = u$. *Then for each* $n \geq N$ *there is an*

$$OD(2^n u;\ 2^n u_1, \ldots, 2^n u_k),$$

where

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Thank you
for
your attention

and

Happy birthday
to
Hadi Kharaghani