Math 4381 - 6378 - Sym

Invariance

We already saw how a circle is invariant under rotation

i.e. $\bar{x} = x\cos\varepsilon - y\sin\varepsilon$

$\bar{y} = x\sin\varepsilon + y\cos\varepsilon$

$$\bar{x}^2 + \bar{y}^2 = r^2 \Rightarrow x^2 + y^2 = r^2$$

So how can these transformations be useful for differential equations

Ex 1 Show

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{x^3 y} + \frac{1}{x^2}$$

is invariant under $\bar{x} = e^{\varepsilon} x$, $\bar{y} = e^{-\varepsilon} y$

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\frac{d}{dx}(e^{-\varepsilon} y)}{\frac{d}{dx}(e^{\varepsilon} x)} = \frac{e^{-\varepsilon}}{e^{\varepsilon}} \frac{dy}{dx} = e^{-2\varepsilon} \frac{dy}{dx}$$

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{y}}{\bar{x}} + \frac{1}{\bar{x}^3\bar{y}} + \frac{1}{\bar{x}^2}$$

$$e^{-2\varepsilon}\frac{dy}{dx} = \frac{e^{-\varepsilon}y}{e^{\varepsilon}x} + \frac{1}{(e^{\varepsilon}x)^3 e^{-\varepsilon}y} + \frac{1}{(e^{\varepsilon}x)^2}$$

$$= e^{-2\varepsilon}\frac{y}{x} + e^{-2\varepsilon}\frac{1}{x^3y} + e^{-2\varepsilon}\frac{1}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{1}{x^3y} + \frac{1}{x^2}$$

Ex2 Show $\frac{dy}{dx} = \frac{y}{x} + \frac{y^3}{x^4}$

is invariant under

$$\bar{x} = \frac{x}{1+\varepsilon x} \qquad \bar{y} = \frac{y}{1+\varepsilon x}$$

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\frac{d}{dx}\bar{y}}{\frac{d}{dx}\bar{x}} = \frac{\frac{(1+\varepsilon x)y' - \varepsilon y}{\cancel{(1+\varepsilon x)^2}}}{\frac{1(1+\varepsilon x) - \varepsilon x}{\cancel{(1+\varepsilon x)^2}}} = (1+\varepsilon x)y' - \varepsilon y$$

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{y}}{\bar{x}} + \frac{\bar{y}^3}{\bar{x}^4}$$

$$(1+\varepsilon x)y' - \varepsilon y = \frac{\frac{y}{1+\varepsilon x}}{\frac{x}{1+\varepsilon x}} + \frac{\left(\frac{y}{1+\varepsilon x}\right)^3}{\left(\frac{x}{1+\varepsilon x}\right)^4}$$

$$= \frac{y}{x} + \frac{y^3(1+\varepsilon x)}{x^4}$$

$$(1+\varepsilon x)\, y' = \frac{y}{x} + \varepsilon y + (1+\varepsilon x)\frac{y^3}{x^4} \qquad \varepsilon y = \frac{\varepsilon x y}{x}$$

$$= (1+\varepsilon x)\frac{y}{x} + (1+\varepsilon x)\frac{y^3}{x^4} \qquad \frac{y}{x} + \frac{\varepsilon x y}{x}$$

$$\Rightarrow \quad y' = \frac{y}{x} + \frac{y^3}{x^4} \quad \checkmark$$

ex. 3 Show

$$\frac{dy}{dx} = \frac{y^2}{xy + x^3}$$

inv $\quad \bar{x} = \dfrac{(y+\varepsilon)x}{y}, \quad \bar{y} = y + \varepsilon$

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\frac{d}{dx}(y+\varepsilon)}{\frac{d}{dx}\left(x+\frac{\varepsilon x}{y}\right)} = \frac{y'}{1+\varepsilon\left(\frac{1\cdot y - xy'}{y^2}\right)}$$

$$= \frac{y^2 y'}{y^2+\varepsilon(y-xy')}$$

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{y}^2}{\bar{x}\bar{y}+\bar{x}^3}$$

$$\frac{y^2 y'}{y^2+\varepsilon(y-xy')} = \frac{(y+\varepsilon)^2}{\frac{(y+\varepsilon)x}{y}\cdot(y+\varepsilon) + \frac{(y+\varepsilon)^3 x^3}{y^3}}$$

$$\frac{\cancel{y^2}\, y'}{y^2+\varepsilon(y-xy')} = \frac{1}{\frac{x}{y} + (y+\varepsilon)\frac{x^3}{y^3}} = \frac{y^{\cancel{3}}}{xy^2+(y+\varepsilon)x^3}$$

$$\{\cancel{y}xy^2+(y+\varepsilon)x^3\}\, y' = y\{y^2+\varepsilon(y-xy')\}$$

$$xy^2y' + (y+\varepsilon)x^3y' = y^3 + \varepsilon y^2 - \varepsilon xyy'$$

$$(xy^2 + yx^3 + \varepsilon x^3 + \varepsilon xy)\, y' = (y+\varepsilon)y^2$$

$$[xy(y+\varepsilon) + x^3(y+\varepsilon)]y' = (y+\varepsilon)y^2$$

$$\cancel{(y+\varepsilon)}(xy+x^3)y' = \cancel{(y+\varepsilon)}y^2$$

$$y' = \frac{y^2}{xy+x^3} \checkmark$$

so

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{x^3 y} + \frac{1}{x^2}$$

invariant under $\bar{x} = e^{\varepsilon}x,\ \bar{y} = e^{-\varepsilon}y$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{y^3}{x^4}$$

invariant under $\bar{x} = \frac{x}{1+\varepsilon x},\ \bar{y} = \frac{y}{1+\varepsilon x}$

$$\frac{dy}{dx} = \frac{y^2}{xy+x^3}$$

invariant under $\bar{x} = \frac{(y+\varepsilon)x}{y},\ \bar{y} = y+\varepsilon$

How is this useful?