# ON THE REPRESENTATION OF QUANTUM FIELD THEORIES OVER A LATTICE

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**Abstract**

A digital model of quantum field theory is presented which describes it as a collection of data-tables having the form $\{X^1, \dots X^n; F^1, \dots F^j\}_i$, where $n, j \in \mathbb{Z}$ and $i$ runs from 1 to infinity. The $X^1, \dots X^n$ are to be taken as $n$-tuples of numbers distributed over a lattice. They constitute the space over which physics is to be represented and define a natural calculus within whose terms that physics can be expressed. An illustration of this model is given in the context of $\varphi^4$ theory. Its implications regarding renormalization and Haag's Theorem are discussed.

## Introduction.

We are accustomed to think of physics as playing out against the backdrop of a continuous manifold of points. For Newton that manifold was Euclidean 3-space – in which resided the stars, planets, and other objects of interest – along with a 1-dimensional continuum, time, with respect to which these objects moved and changed. The ground-breaking work of Einstein and Minkowski replaced this picture with that of spacetime, a pseudo-Riemannian 4-manifold. Einstein's subsequent demonstration that gravity could be understood as an expression of the geometry of this manifold, and that that geometry is, itself, an active participant in physics, can only have reinforced the conviction that such manifolds are, indeed, the proper venue for physics. And we are visual creatures; our eyes and brains present us with a picture of the world that *looks* like a continuous manifold. So it is, perhaps, inevitable that we would want to represent physics to ourselves in such a way.

This view has not escaped criticism. Kant argued, very insightfully, that space and time, rather than being true physical primitives, are modes of perception characteristic of the human mind; they are, in his opinion, categories imposed by the mind upon reality in the act of experiencing it. (Although this idea tells us quite a bit about what reality isn't, it does not have much to say about what reality is.) More recently there have been attempts made to accommodate physics within various information-theoretic frameworks [1]. I will not review these but will only call attention to a few ideas that are relevant to the subject under our consideration. One of these is the Virtual World Simulation Hypothesis [2], advanced by Bostrom. According to this notion we and our reality are being simulated by a "computer" (or some equivalent form of information-processing system) in much the same way that 'Sims' are simulated in the eponymous game. Of course, our simulation is far more complicated than that of the 'Sims' – so much so that we cannot even tell that we are 'Sims.' Bostrom's idea raises a number of provocative questions. For one thing, we must wonder whether a 'Sim' *could* be sentient. Lacking any profound insight into the metaphysics of consciousness, we should probably just admit that we don't know. At least, there is no very obvious reason why it couldn't be. For another thing, we must wonder about the "computer" and the world it is imagined to exist in. Are these things real or just useful fictions? If real, what are they? For more immediate purposes, another question comes to mind – how *could* something like a computer represent a universe?

## The Universe, Analogue or Digital?

Most digital interpretations of physics assign the values of their physical variables to a discreet, denumerably infinite, lattice of spacetime points. To represent a *continuous* manifold in this way would require $\aleph_1$ addresses at which to place this information which is, of course, not possible. Indeed, we may state, as a simple principle of digital physics:

*The universe can be represented as an infinite, denumerable, sequence of symbols drawn from a finite list.*

That is to say, everything we could ever experience in, or want to know about reality, can be represented as discreet bits of information. It is not necessary to propose any "computer," existing in a separate world, doing anything resembling calculations. Nor is it necessary to propose any separate world. Neither will we propose any "time" different from the ordinary coordinate time that Einstein has already argued away as subjective. Specifically:

*The universe can be represented as a collection of $\aleph_0$-infinite data-tables, each reading $\{X^1, \ldots X^n; F^1, \ldots F^j\}_i$ where n denotes the number of dimensions of the space in which these addresses are to be placed, j the number of physical variables of interest, and i runs from 1 to $\infty$*

For the above description to be specific there must exist an algorithm to which these quantities and symbols conform. This algorithm should be a thing that is also representable in a digital way. A "physical variable of interest" – call it $F^1$ - might be a real scalar function. It might be a real covarient $4-$vector $B_i$ (in which case it would consist of four real numbers). It might be a complex scalar field, $\varphi$, consisting (in effect) of two real numbers. It might be an operator that takes states (which there must be ways of specifying) into other states. It might be almost any kind of mathematical object and we will not try to be too specific here. That any of the $X^i$ or $F^j$ might be infinite poses a problem; it is unclear how an information processing system could represent this situation. We will stipulate that no coordinate address or physical variable can ever take on an infinite value.

These numbers and other mathematical objects, $X^i$ and $F^j$, are the actual, irreducible, "stuff" that makes up physics – they are absolute and real. There is nothing arbitrary about this sequence of 1s and 0s and other useful symbols. It comprises the information that *is* our world. The means of its representation may appear arbitrary. But it is real all the same. This view is not unlike that expressed by Kleinert in his World-Crystal model [3].

As mentioned, we take an Einsteinian approach to time. That is to say, time is merely another physical coordinate. It does not "change" or "move" or do anything different from the theory's three spatial dimensions. The past, present, and future simply *are* in this theory – no different from the situation that obtains in ordinary relativistic physics. Those numbers that constitute the data-table that is our universe, and the algorithm they conform to, must be regarded as facts independent of space and time. In this sense the universe is best looked at as an "already-calculated" mathematical structure.

## Regarding the Representability and Parsimony of Information.

While Newton could express physics in the language of differential equations, we enjoy no such privilege. Newton's mathematics exists only within a continuous manifold. We have to specify, explicitly, what is meant by "derivative" or "integral" in terms limited to numerical operations such as addition and multiplication and the operations of logical selection – the sorts of operations that information processing systems are suited to performing. I cannot ask a computer to differentiate the function $a^2$. The computer has no natural notion of a function and no idea what I could possibly mean by "differentiate." I can, however, tell the computer to construct a data-table consisting of $10^6$ pairs of real numbers, $\{a, a$ times $a\}$ where $a$ is between 0 and 1. Picking out a particular value of $a$, $a_0$, I can tell the computer to find the $a$ closest in value to $a_0$, subtract $a_0$ times $a_0$ from that $a$ times itself, then divide the result by the difference between $a$ and $a_0$. The result would be the derivative in question to a high degree of accuracy. In this way differentiation becomes a representable process. It becomes a prescription telling an information processing system how to treat a defined set of numerical values. All the mathematical operations available to a proper digital theory must be specifiable in such a way.

One should ask what else needs to be specified – what information has the world to consist of – for things to appear to us in the way that they do? And is it desirable that a theory require "more" information than is needed to describe our world? In perceiving an image displayed on a computer screen, or printed on paper, one is usually looking at a large collection of finely spaced dots with nothing in between them. But they appear as continuous lines, letters, and shapes. The eye is not aware of the empty space between the many dots. And there is no reason for it to be – this space contains nothing relevant or necessary to the image in question. Digital physics, for our purposes anyway, suggests that reality works in a like fashion. For it to work otherwise, for it to be as the continuum picture describes it, burdens physics with an increased informational cost while affording it nothing in terms of increased explanatory power.

## Hypercubic Lattices and Isotropy.

When solving a partial differential equation over a Cartesian lattice with a constant spacing $\epsilon$ it is convenient to approximate the derivatives of interest as:

1) $\partial_{x_i} f[x_i] = (f[x_{i+1}] - f[x_{i-1}]) / 2\,\epsilon$ ,

2) $\partial_{x_i,x_i} f[x_i] = (f[x_{i+1}] + f[x_{i-1}] - 2\, f[x_i]) / \epsilon^2$, and so forth.

Some simple, and physically important, partial differential equations lend themselves to very straightforward solution over such a lattice. Consider a field, $\varphi[x_i]$, satisfying the Klein-Gordon equation (we work with just two dimensions here, $x$ and $t$):

3) $(\Box^2 + m^2)\, \varphi[x_i] = 0$ .

Writing $\varphi[x_i] = e^{i(E\,t_i - k\,x_i)}$, we find the dispersion relation:

4) $\cos(\epsilon E) = \cos(\epsilon k) - \epsilon^2 m^2/2$ .

For small energies, or small $\epsilon$, this dispersion relation approximates that of the familiar continuum solutions. As $k$ approaches $\pi/\epsilon$ this relation changes and the "energy" of this plane-wave solution diverges (if $m \neq 0$) then turns downward – essentially, the solution duplicates itself in the other direction as $k$ exceeds its critical value. This phenomenon defines the so-called Brillouin zones for this lattice, given equation 3). Thus the structure of the lattice, and the physical nature of the equation, impose a kind of upper limit on the momenta allowed to particles and waves represented thereupon. When we consider the same situation in the context of Minkowski spacetime we see that this upper limit is direction-dependent. If we consider a plane-wave propagating in the (1,1,1) spatial direction we find that the separation of lattice points (in that direction of propagation) is increased by a factor of $3^{1/2}$ relative to what it would be if the wave were propagating in the *x*, *y*, or *z* directions. Models of this kind do not respect Lorentz or Poincaré invariance; one frame, moving, or angled, with respect to another will not, in most cases, seem to contain the same physics as would be observed in the other since the lattice over which its physical observables are defined will differ in structure. This would, if $\epsilon$ were large enough, result in observable anisotropies in the cosmic-ray energy spectrum, as has been pointed out by Savage, et. al. [4].

We will adopt the symbol $\Delta_{x_i}$ to designate the process embodied in Eq 1), $\Delta_{x_i, x_i}$ to designate that of Eq 2), etc. This new $\boldsymbol{\Delta}$ operator is linear (as is conventional differentiation). I.e. $\boldsymbol{\Delta}_x (f + g) = \boldsymbol{\Delta}_x f + \boldsymbol{\Delta}_x g$. $\boldsymbol{\Delta}_x (f\ g)$ will not, however, equal $g\,\boldsymbol{\Delta}_x f + f\,\boldsymbol{\Delta}_x g$, under most circumstances. As a matter of nomenclature, we call the coordinate addresses of our lattice points $X^\mu[i]$ or, simply, $X^\mu$. We will denote the lattice L.

Having adopted a hypercubic lattice as our computational framework it is necessary to define a metric upon it – although it is not quite correct to describe such a thing as a metric. A metric defines distance according to $ds^2 = \boldsymbol{g}_{ij}\, dx^i dx^j$. But infinitesimals cannot be employed here. We will, instead, consider $\boldsymbol{g}_{ij}$ to be a mathematical object that applies only at the coordinate address where it is defined and gives us the physical distance between that address and its nearest coordinate-neighbors – that distance to be calculated as $\sqrt{\boldsymbol{g}_{ij}\, \Delta x^i\, \Delta x^j}$ where $\Delta x^i$ designates the coordinate distance between that address and the neighbor of interest.

This "metric," $\boldsymbol{g}_{ij}$, raises and lowers the indecise that appear in the equations that constitute physics. Higher-order derivatives of $f$ may be obtained by simply regarding lower-order derivatives as new fields, defined over the lattice points, and redifferentiating them in the manner already specified. This "metric" allows for the definition of past and future light-cones throughout the lattice. Also, $\boldsymbol{g}_{ij}$ permits the definition of an operation analogous to Newtonian integration. If we consider a function, $f$, defined over a region $R$, we may consider its integral to be:

5) $\int_R f\, dV = \epsilon^N \sum_{X^\mu \in R} \sqrt{\left| \det \boldsymbol{g}_{ij}\, [X^i] \right|}\, f[X^i]$ where $N$ denotes the dimensionality of our space (here 4).

Were we to consider general relativity we would take $\boldsymbol{g}_{ij}$ to be another physical variable of interest. But we will not be doing that in this article. So, since we are employing a hypercubic lattice, we take $\boldsymbol{g}_{ij}$ to be the usual Minkowski metric $\boldsymbol{\eta}_{ij}$.

## Coordinate Systems.

If the world consisted only of electromagnetic waves propagating within an empty background "spacetime," we could call $F^1$ by another name, $A_\mu$ (the 4-potential), and just worry about how to associate its values with those of the coordinate addresses defined by our lattice. We require that such an electromagnetic wave, expressed in Feynman gauge, satisfy $\boldsymbol{g}^{ij}\, \boldsymbol{\Delta}_{ij}\, A_\mu = 0$. (In effect, this field's components satisfy 3) with $m$ set to 0.) This gives us a field that looks very much like a classical electromagnetic wave (as long as its wavelength is not so short as to approach $\epsilon$).

Other writers have noted that "time is defined so that motion appears simple [5]." We will take this principle a step further: Spacetime will be defined so that *the motion of light* appears simple. The points comprising our lattice are real, numerical, coordinate addresses. But there is no way for us to determine the actual numbers used to designate them. Neither can we know the numerical values assigned to the physical fields of interest – we can't know what system of units Nature employs. If we wish to do physics, we will have to invent these things for ourselves.

While our theory does not yet provide for observers, we will cheat a little and imagine that there are, somehow, physicists inhabiting this world. These physicists are very clever and will figure out that they, too, can assign quartets of real numbers to the lattice points. Since we assume that they want to make their work as easy as possible, we can imagine them, acting rather like an infinite team of surveyors, using light rays to map out and assign to these various points (the only places where observations can be made) their own quartets of coordinate addresses. Unlike the coordinate addresses of the lattice, these addresses *are* arbitrary; they are up to the whims of the physicists. These physicists will, probably, want to demand of their coordinate system that $\boldsymbol{g}_{\text{ij}} = \boldsymbol{\eta}_{\text{ij}}$; this is the just simplest way for them to do things. In effect, they will have defined new physical fields, call them $x^\mu[i]$, or $F^2$ if you prefer, which depend on the $X^\mu[i]$. Now, there are infinitely many ways in which they can to do this. Suppose that a second group of physicists relabels our lattice points with a different coordinate system, $\overset{-}{x}^\mu[i]$, which also has $\boldsymbol{g}_{\text{ij}} = \boldsymbol{\eta}_{\text{ij}}$. The overall transformation $\overset{-}{x}^\mu[i]$ $(\to X^\mu[i]) \to x^\mu[i]$ is what functions as a Lorentz (or Poincaré) transformation in the context of this theory [6]. This procedure makes sense here only because of our choice, $m = 0$; this implies that $|E| = |\boldsymbol{k}|$, allowing us to define something like a "speed of light." But, in a general sense, this theory is not Lorentz invariant. Having specified a lattice over which to define the process of differentiation we have, in effect, defined an absolute rest-frame. Physics will not look the same to observers moving relative to it. We will revisit this matter in greater detail below.

## Towards A Realistic Model of the Universe?

What should be taken as "functions of physical interest?" One might try to represent something quite simple – maybe the non-relativistic Schrodinger equation or something from classical physics – over such a lattice, just to see what happens. But this would lead to no useful conclusions. While we do not know what distance-scale this theory operates over, it certainly falls far below anything that could be called non-quantum mechanical or non-relativistic. It would be much better to take relativistic quantum field theory (QFT) as the model. One could try to be realistic and take the Standard Model as a template. One could be ambitious and consider something more speculative. But let us take the simplest case as an example. We will consider a real, self-interacting, scalar field, $\varphi$. While such a model does not describe actual physics, it does contain many of the elements we would expect to find in a theory that might, ultimately, do so. The Lagrangian for this field is written as:

6) $\mathfrak{L}[\varphi] = \frac{1}{2}\left(\boldsymbol{\Delta}^\mu \varphi \boldsymbol{\Delta}_\mu \varphi - m_0{}^2 \varphi^2 - \frac{\lambda_0}{4!}\varphi^4\right)$, where $\lambda_0$ is assumed to be positive (ruling out bound states) and small (so that perturbation theory can be employed).

QFTs can be represented in a number of different, yet mathematically equivalent, ways. The Heisenberg Picture, and the Interaction Picture that supervenes upon it, have the distinct advantage of avoiding the use of functionals of $\varphi$ in their mathematics. The employment of such functionals is problematic since it is not clear how we could confine $\varphi$ to a denumerably infinite domain of possibilities in any natural or convenient way. We will, therefore, assume that our hypothetical information processing system utilizes something like the Interaction Picture in doing its work.

Ignoring $\frac{\lambda_0}{4!}\varphi^4$, we would, at least in the continuum limit, have an easy problem to solve. Requiring that $\varphi$ should extremize the action corresponding to $\mathfrak{L}_0$ (by which is meant the above Lagrangian without the interaction term) we would obtain (using standard normalization conventions):

7) $\varphi[x] = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_k}} (e^{-i k x} a_k + e^{i k x} a_k{}^\dagger)$ (where $\omega_k = k_0 = \sqrt{\boldsymbol{k}^2 + m_0{}^2}$ ).

Were we only solving a partial differential equation the $a_k$ would be taken as complex numbers. Instead, we interpret $\varphi$ as an operator which acts upon a basis of states. We define its canonical field momentum, $\pi[x]$, and demand that that satisfy the equal-time commutation relations $[\varphi[\boldsymbol{x}, t], \pi[\boldsymbol{y}, t]] = i\,\delta^3[\boldsymbol{x} - \boldsymbol{y}]$. In this way we arrive at the commutation relations:

8) $[a_{k_1}, a_{k_2}] = [a_{k_1}{}^\dagger, a_{k_2}{}^\dagger] = 0,$
$[a_{k_1}, a_{k_2}{}^\dagger] = \delta_{k_1 k_2}$ .

It will be noted that $k$ is treated as a discreet variable. This is necessary if we are to work within a picture of physics as denumerable tables of information. We restrict $k$ to a denumerable set and call that set $K$.

Solutions such as $e^{-i k x}$ make sense in the continuum limit but can only be useful as a guide here. We will still demand of $\varphi$ that it extremize $\mathfrak{L}_0$ (with integration being understood in the sense of Eq 5)). Since $\mathfrak{L}_0$ involves $\varphi$[x] only to second order we should be able to write:

7') $\varphi[x] = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k} \in K} (s[\boldsymbol{k}, x]\, a_k + s^*[\boldsymbol{k}, x]\, a_k{}^\dagger)$ .

But it would be incorrect to assume that these $s[\boldsymbol{k}, x]$ satisfy Eq 3). The Klein-Gordon equation derives from $\mathfrak{L}_0$ through the classical procedure of Euler and Lagrange. This procedure employs integration by parts which does not work here. Therefore Eq 4) is not usable. While we cannot exactly solve the problem of extremizing $\mathfrak{L}_0$ over our lattice, we can employ a simple trick to do something just as good. We know that, in the continuum limit, the extremal solutions can be expressed as $e^{\pm i k x}$. We know that they always satisfy $(\partial_t \varphi)^2 - (\partial_x \varphi)^2 - (\partial_y \varphi)^2 - (\partial_z \varphi)^2 + m_0{}^2 \varphi^2 = 0$. The foregoing terms are exactly those that appear in $\mathfrak{L}_0$. It only remains to replace the $\partial$s with $\Delta$s, since these latter represent the values *actually* used in our calculations. Taking $\varphi[X^\mu[\text{i}]] = \exp[-i\,(\omega_{\boldsymbol{k}} X^0[\text{i}] - \boldsymbol{k}^j X^j[\text{i}])]$ with (j = 1,2,3) we obtain the dispersion relation:

9) $\sin^2[\epsilon\,\omega_{\boldsymbol{k}}] = \sum_i \sin^2[\epsilon\,\boldsymbol{k}^i] + \epsilon^2 m_0{}^2$ .

Accordingly, we set:

10) $\varphi[X^{\mu}[\text{i}]] = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k} \in \mathrm{K}} (s[\boldsymbol{k}, X^{\mu}[i]]\, a_{\boldsymbol{k}} + s^{*}[\boldsymbol{k}, X^{\mu}[i]]\, a_{\boldsymbol{k}}^{\dagger})$ with

11) $s[\boldsymbol{k}, X^{\mu}[i]] = \frac{1}{\sqrt{2\,\omega_{\boldsymbol{k}}}} \exp[-\, i\, (\omega_{\boldsymbol{k}}\, X^{0}[\text{i}] - \boldsymbol{k}^{j} X^{j}[\text{i}])]$ , $V$ being the volume of our 3-space, and $\omega_{\boldsymbol{k}}$ defined by Eq 9) with $\boldsymbol{k} \in \mathrm{K}$.

To confine $\boldsymbol{k}$ to a denumerable set of values we impose periodic boundary conditions on $X^{\boldsymbol{i}}[i]$, restricting the coordinate addresses $X^1[i]$, $X^2[i]$, and $X^3[i]$ to values between $L$ and $-L$, and reconfiguring $\boldsymbol{g}_{\text{ij}}$ such that it recognizes $L$ and $-L$ as synonymous. In effect, the three ends of the spatial part of our lattice are stitched together producing a 3-torus. And we require that $\varphi$ be, likewise, periodic. Now Eq 9) makes no sense when $\epsilon^2 m_0{}^2$ approaches unity so we will, instead, take this to be a small quantity. Effectively, $|\boldsymbol{k}|$ cuts off at very nearly $\pi/(2\,\epsilon)$ (at which point $\omega_{\boldsymbol{k}}$ reaches its maximal value). This implies that K is not only denumerable, but *finite*. It also exhibits anisotropy owing to the boudary conditions imposed on L.

Because K is finite the $s[\boldsymbol{k}, X^{\mu}[i]]$ and $s^{*}[\boldsymbol{k}, X^{\mu}[i]]$ do not constitute a complete, orthonormal, set of functions. The above-mentioned equal-time commutation relations will no longer give us Eqns 8) in any exact sense. We must therefore *postulate* Eqns 8) to be a part of our algorithm; they are to be regarded as true and not dependent on anything else. It will also be observed that $\varphi$ no longer strictly obeys the principle of microcausality – $[\varphi[X^{\mu}[\text{i}]], \varphi[X^{\mu}[\text{j}]]] \neq 0$ for $X^{\mu}[\text{i}]$ and $X^{\mu}[\text{j}]$ space-like separated (although it is very nearly satisfied for separations significantly greater than $\epsilon$).

The creation operators $a_k{}^{\dagger}$construct, from an assumed vacuum state $|0\rangle$ (that satisfies $a_k|0\rangle = 0$ for all $\boldsymbol{k} \in \mathrm{K}$), a Fock space of multiparticle states. A vector in this space, here designated $|\psi\rangle$, is to be regarded as a possible state of the universe. It is this vector that describes the world that observers would see around themselves. The other variable of physical interest, $\varphi$, is not directly observable. Instead, it serves to define, and to guide, the evolution of, $|\psi\rangle$. (We assume $\langle\psi|\psi\rangle = 1$.)

By ignoring the $\frac{\lambda_0}{4!}\varphi^4$ term, one arrives at a theory of free scalar particles – a good thing since observation suggests that free particles do exist as legitimate building blocks of nature. Taking account of $\frac{\lambda_0}{4!}\varphi^4$, the Interaction Picture regards the vector $|\psi\rangle$ as a function of time (defined here as $X^0[\text{i}]$) which evolves according to:

12) $i\, \boldsymbol{\Delta}_{X^0}\, |\psi(X^0)\rangle = \mathrm{H}_I\, |\psi(X^0)\rangle$ (where $\mathrm{H}_I$ designates $\int_V \frac{\lambda_0}{4!}\varphi^4[X^{\mu}]\, d^3 X$ and time is represented by $X^0$).

So physics, as described by this model, consists of *two* data-tables defined in tandem with one another. One of these is represented by $\{X^0, X^1, X^2, X^3; \varphi\}_i$ and the other by $\{X^0;\ |\psi\rangle\}_i$ where $i$ runs from 1 to $\infty$ $\varphi$ represents an operator and $|\psi\rangle$ represents a vector in the Fock space constructed from $|0\rangle$ using its creation operators $a_k{}^{\dagger}(k \in \mathrm{K})$. This way of doing things is general. (Had we chosen QED as our example we would have ended up with three data-tables, $\{X^0, X^1, X^2, X^3; \Psi\}_i$ , $\{X^0, X^1, X^2, X^3; A_\mu\}_i$ , and $\{X^0;\ |\psi\rangle\}_i$ where $\Psi$ denotes the Dirac electron/positron field, $A_\mu$ the photon field, and $|\psi\rangle$ a vector in the Fock space constructed from the vacuum using the creation operators defined by these two fields.)

## Renormalization and Scattering.

The above scheme, as it applies to $\{X^0, X^1, X^2, X^3; \varphi\}_i$, provides a background of free particle states in terms of which $|\psi>$ is defined. The dependence of this vector on $X^0$ is governed by Eq 12), the solution of which is described in standard QFT texts. Because $\lambda_0$ is small, Eq 12) can be solved using perturbation theory. Doing so yields a series of terms, each corresponding to a Feynman diagram of $\varphi^4$ theory. (The fact that we are using $\mathbf{\Delta}_{X^0}$ instead of $\partial_{X^0}$ would not be expected to lead to very significant differences here.) These terms are simplified by means of Wick's theorem. This theorem depends only on Eqns 8) and is as applicable here as it would be in the continuum picture. The resulting contractions, $<0|\mathrm{T}(\varphi[x], \varphi[y])|0>$, are evaluated by means of the contour integral $\exp[-i\,\omega_{\boldsymbol{k}}\,|\,t\,|] = \frac{i\,\omega_{\boldsymbol{k}}}{\pi}\int_{-\infty}^{\infty} \frac{\exp[i\,k_0\,t]}{k_0{}^2 - \omega_k{}^2 + i\,\eta}\, dk_0$ $(\eta \to +0)$ something like which ought to still work in our case. (We have not restricted $k_0$ to finite values. Essentially, we would replace the integral with a summation over $\boldsymbol{k_0}$.)

As $\epsilon \to 0$ those terms with internal loops give rise to integrals that are UV divergent over their internal momenta – something taken to indicate trouble with the theory at very small distance scales. No approximations are made in the course of deriving this result, and it is mathematically robust. Our theory would deliver a similar one except that it possesses a natural high-momentum cutoff ($\Lambda \approx \pi/2\epsilon$) – distances cannot be arbitrarily small here. We would, rather, expect a result very much like that obtained by applying this cutoff to the integrals derived in the continuum limit. Thus, while an artifice introduced to render integrals finite in the continuum model, $\Lambda$ becomes an advantageous result of the present one. Owing to the renormalizability of $\varphi^4$ theory it is possible to subsume the contributions from these loop diagrams into redefined values for the observed mass, coupling constant, and wavefunction normalization of the theory. Specifically, if we had begun by defining the Lagrangian as:

6') $$\mathfrak{L}_R[\varphi] = \frac{1}{2}\left( f\left(m_{\text{phys}}, \lambda_{\text{phys}}, \Lambda\right) \mathbf{\Delta}^\mu\, \varphi\, \mathbf{\Delta}_\mu\, \varphi \;-\; g\left(m_{\text{phys}}, \lambda_{\text{phys}}, \Lambda\right) \varphi^2 \;-\; \frac{h\,(m_{\text{phys}}, \lambda_{\text{phys}}, \Lambda)}{4!}\, \varphi^4 \right)$$

we would end up with a theory that consisted of particles with mass $m_{\text{phys}}$ interacting with a coupling constant $\lambda_{\text{phys}}$ only through terms that are finite. The exact numerical values of $f$, $g$, and $h$ depend upon the manner in which the cutoff is performed [7]. Regardless of how it is performed, it should provide an realistic estimate of what would obtain if the above theory were actually to be calculated out. So there ought to be a set of numbers $f$, $g$, and $h$ consistent with the world our observers experience.

We can now return our attention to the invariance properties of this model and assume renormalization to have been performed by choosing the required values for $f$, $g$, and $h$ above. Any scattering event will correspond to a matrix element proportional to $\sum_{X^\mu \in L} \exp\left[i\left(\sum k_\mu^{\text{out}} - \sum k_\mu^{\text{in}}\right) X^\mu\right]$ so $k^\mu$ is a conserved quantity here (with $\sin^2\,[\epsilon\, k^0] = \sum_i \sin^2\left[\epsilon\,\boldsymbol{k}^i\right] + \epsilon^2 m_{\text{phys}}{}^2$ ) and $\boldsymbol{k}^i \in$ K. But $k^\mu$ is not Lorentz invariant – observers moving relative to L, who perform a Lorentz transformation on $k^\mu$, will find that their new vector no longer satisfies the above dispersion relation. This poses no real problem as the scattering processes they observe are "precalculated" in the rest-frame defined by L. And, since Lorentz transformations are linear, all observers will have to agree that $\sum k^\mu$ is conserved. It, actually, reduces to something of a nomenclatural issue. (What do the moving observers want to call $k^\mu$?) And it only becomes a matter of concern when momenta approach $|\boldsymbol{k}| \approx \pi/2\epsilon$. For smaller momenta (relative to L) this theory is, for all intents and purposes, Lorentz invariant. An important exception occurs in the case where $m_{\text{phys}} = 0$. Here something like true Lorentz invariance is regained. (These massless particles can be considered analogous to the "photons" used by our hypothetical physicists in the construction of their coordinate systems.) Still, the momenta allowed to these particles cuts off at $\pi/2\epsilon$ [8].

## Haag's Theorem.

An important no-go theorem, due to R. Haag [9], casts doubt on the mathematical consistency of all QFTs. Adapted to our terminology it implies that, given a set of operators satisfying 8), which also possess a vacuum state from which we can construct a Fock space, we can assemble infinitely many other such set of operators (also satisfying 8)) for which no vacuum state, constructible from our original operators, can exist. These sets of operators will not be unitarily equivalent to our original choice and the physics deriving from 8) must, therefore, yield ambiguous results. Haag's argument centers on the infinitely many degrees of freedom assumed in QFTs. After assuring us of the mathematical soundness of theories where the degrees of freedom are finite, he goes on to state: *"If we pass now to the limit $N \to \infty$ one new feature appears. A possible basis vector results from any distribution of integer numbers $\nu_k$ over the infinitely many oscillators. The "number' of these possibilities is no longer countable. It is given by $\aleph_0^{\aleph_0} = \aleph_1$."* He concludes: *"The point is, however, that for infinite $N$, (14) is no longer a consequence of (12). In other words, there will be different irreducible representations of (12)."* (By *(12)* Haag means our commutation relations 8). By *(14)* he means $a_k|\psi_0> = 0$, defining the existence of a unique vacuum state.) But, in our theory, $K$ is finite; Haag's argument cannot go forward.

We have not, of course, proven that this theory *is* mathematically consistent. We have only shown that it is not inconsistent by Haag's line of reasoning. And it is interesting to reflect that the thing that makes QFTs inconsistent according to Haag's theorem is the same thing that makes them non-representable digitally.

## Conclusion.

The example of a simple QFT has been realized over a hyperbubic lattice L. To do this in a way that is calculable we have introduced two data-tables – $\{X^0, X^1, X^2, X^3; \varphi\}_i$ and $\{X^0; |\psi>\}_i$. And we have specified algorithms to which these data-tables must conform (these employing only simple logical operations and the notion of differentiation implied by the lattice). It has been shown that the structure of this lattice imposes a momentum cutoff at, approximately, $\pi/2\epsilon$. It is argued that this (naturally dictated) cutoff provides an effective regularization, allowing this QFT to be renormalized with all (formerly) divergent Feynman diagrams absorbed into the values $m_{\text{phys}}$ and $\lambda_{\text{phys}}$. Infinities are never involved here.

Dirac's Interaction Picture has been used in formulating the above approach. But the mathematical integrity of this approach, and that of QFT generally, is called into question by Haag's theorem. In order to make the Fock space in which $|\psi>$ is represented separable we have introduced periodic boundary conditions on L and $\varphi$. This has the effect of rendering $K$, the set of allowed momenta, finite, circumventing the Haag proof.

Now we are left with a curious-looking picture of physics. An observer moving very fast with respect to the rest-frame defined by L will see an anisotropically distributed set of momenta available to his particles. (Whatever "Lorentz invariance" he sees in this theory derives only from the (formal) Lorentz invariance of the algorithm defining the data-tables.) But this need not be viewed as a problem. (Instead, it offers opportunities for the experimental verification of our idea.) Since we see no obvious anisotropy in the momenta of particles around us, or in our accelerators, we may assume that the rest-frame defined by L is somewhat close to the cosmic rest-frame (i.e. to our rest-frame).

The subject of gauge invariance has not been discussed; in connection with $\varphi^4$ theory, it does not arise. But all realistic QFTs seem to display this property in one form or another. So it should be commented upon. Suppose that we consider QED. The values of $A_i$ and $\Psi$ that appear in their relevant data-tables are fixed and absolute – this theory does (and must) operate within one fixed choice of gauge. But what this is we cannot ever determine. Really, it makes no difference. Our observers may relabel $A_i$ and $\Psi$ using whatever choice of gauge they desire and physics will still work just fine for them. This is in no way a consequence of our theory. It is, rather, a consequence of the nature of their QFT. Why realistic QFTs possess this property is an interesting question, but one lying outside the scope of this article. It is better to be explored in work to follow where $\boldsymbol{g}_{\text{ij}}$ is considered a function of physical interest, defined over L.

## References and Footnotes.

1) For a general review see Wikipedia: Digital Physics.

2) Bostrom, Nick. *Philosophical Quarterly*, 2003, Vol. 53 No.211, 243.

3) Jizba, P.; Kleinert, H.; Scardigli, F. *Physical Review D* **81** (2010), 084030 and references cited therein.

4) Beane, S .R., Davoudi, Z, Savage, M. J., arXiv:1210.1847 [hep-ph].

5) Misner, C. W.; Thorne, K. S.; Wheeler, J. A. *Gravitation*, p. 23, W. H. Freeman and Company (1971).

6) Of course, these physicists are not obliged to choose any particular kind of coordinate system. Should they prefer one that is not rectilinear, their metric will still satisfy $\boldsymbol{\eta}_{\text{ij}} = \boldsymbol{\Delta}_{X^i}\, x^{\mu}\; \boldsymbol{\Delta}_{X^j}\, x^{\mu}\; \boldsymbol{g}_{\mu\nu}$. But they will have to adjust their processes of differentiation to accommodate the requirements of differential geometry.

7) One method is to Wick-rotate the denominators of the propagator terms and transform their 4-momentata into hyperspherical coordinates, integrating out to the radial distance $\Lambda$. The values of $f$, $g$, and $h$ obtained by employing this method follow from Cline, J. M. http://www.physics.mcgill.ca/~jcline/qft1b.pdf and should correspond, roughly, to those appropriate to our model.

8) For an interesting discussion of Lorentz "invariance" when $E^2 \neq \boldsymbol{p}^2 + m^2$ see Magueijo, J;, Smolin, L. *Physical Review D* **67** (2003), 044017.

9) Haag, R. *Mathematisk-fysiske Meddelelser*, **29**, 12 (1955). (See also Hall, D: Wightman, A. S. *Mathematisk-fysiske Meddelelser*, **31**, 1 (1957).)