Math 4315 - PDE's

Method of characteristic (M of C)

$$a u_x + b u_y = c$$

M of C $\quad \dfrac{dx}{a} = \dfrac{dy}{b} = \dfrac{du}{c}$ $\leftarrow$ solve in pairs

Ex 1 $\quad u_x - u_y = 2x \qquad u(x, x) = 1$

$$\frac{dx}{1} = -\frac{dy}{1} = \frac{du}{2x}$$

(1) $dx = -dy \Rightarrow x + y = c_1$

2) $2x\,dx = du \Rightarrow u - x^2 = c_2$

Sol^n $\quad u - x^2 = f(x+y) \quad$ or $\quad u = x^2 + f(x+y)$

Now the B.C. $\quad u(x, x) = 1 \Rightarrow 1 = x^2 + f(2x)$

let $\lambda = 2x$ so $x = \frac{\lambda}{2}$

$$1 = \left(\frac{\lambda}{2}\right)^2 + f(\lambda) \Rightarrow f(\lambda) = 1 - \frac{\lambda^2}{4}$$

Now back to PDE

$$u = x^2 + f(x+y)$$

$$= x^2 + 1 - \frac{(x+y)^2}{4}$$

Ex2 $\quad x u_x + (x+y) u_y = u \qquad u(x,0) = x^2$

M of C $\quad \frac{dx}{x} = \frac{dy}{x+y} = \frac{du}{u}$

1^{st} pair $\quad \frac{dx}{x} = \frac{dy}{x+y}$ or $\frac{dy}{dx} = \frac{x+y}{x}$ Linear Homo

or $\frac{dy}{dx} - \frac{y}{x} = 1 \qquad \mu = e^{-\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x} \Rightarrow \frac{y}{x} = \ln x + c_1$$

so $c_1 = \frac{y}{x} - \ln x$

Next $\frac{dx}{x} = \frac{du}{u} \Rightarrow \ln|x| = \ln|u| - \ln C_2$

so $\ln C_2 = \ln|u| - \ln|x|$

$$\Rightarrow C_2 = \frac{u}{x}$$

sol^n $c_2 = f(c_1) \Rightarrow \frac{u}{x} = f\left(\frac{y}{x} - \ln x\right)$

sub in B.C. $\frac{x^2}{x} = f(0 - \ln x)$

$$\boxed{\text{sol}^n \quad u = x^2 e^{-y/x}}$$

let $-\ln x = \lambda \Rightarrow x = e^{-\lambda}$

$f(\lambda) = e^{-\lambda}$ sol^n $\frac{u}{x} = e^{-(y/x - \ln x)} = x e^{-y/x}$

$xyu_x + yuu_y = u^2$ ✓ $\quad u(x,1) = x$

$$\frac{dx}{xy} = \frac{dy}{yu} = \frac{du}{u^2} \qquad \frac{dy}{y} = \frac{du}{u} \qquad \frac{u}{y} = c_1$$

$$\frac{dx}{x} = \frac{dy}{u} = \frac{dy}{c_1 y} \qquad c_1 \ln x = \ln y$$

$$\frac{u}{y}\ln x - \ln y \qquad\qquad u(x,1) = x$$

$$\frac{u}{y}\ln x - \ln y = f\left(\frac{u}{y}\right) \qquad \cancel{f(x) = x\ln x}$$

$$u \ln x - \cancel{\ln 1} = f(u)$$

$$\frac{u}{y}\ln x - \ln y = \frac{u}{y}\ln\left(\frac{u}{y}\right)$$

$$\frac{u}{y}\ln\left(\frac{xy}{u}\right) = \ln y$$