18 April 2011

The Macdowell-Mansouri Extension

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I. Introduction; First Order Gravity

MacDowell-Mansouri gravity is an extension of the first-order formalism for gravity that goes back to the days of Cartan, and which is demanded when Dirac particles are coupled to gravity. In the standard first-order formalism, the basic variables are a vierbein e_μ^A and a connection ω_μ^{AB}. Each is a vector in (3 + 1) spacetime. But these quantities also live in an internal O(3,1) space. In that internal space the vierbein e transforms as a vector and the connection ω transforms in the adjoint representation, as an antisymmetric tensor. I will throughout this note contract the vierbein e into a gamma matrix, and the connection ω into a pair of gamma matrices:

$$e \equiv e_\mu^A \gamma_A \qquad\qquad \omega \equiv \frac{1}{2}\omega_\mu^{AB}\gamma_A\gamma_B$$

Evidently these gamma matrices live only in the internal space, and have no spacetime dependence.

There is an action (often called Palatini, which will be written down later) which describes this version of gravity. Variation of this action with respect to the 24 connection variables ω (up to now taken to be general) gives a set of algebraic equations which can be solved in terms of the vierbeins and their spacetime gradients. The vierbein is related by definition to the metric tensor

$$g_{\mu\nu} = \frac{1}{2}\{e_\mu, e_\nu\}$$

And the net result of the 24 "equations of motion" for the connection variables ω is to specify that they are Levi-Civita, i.e. described by the Christoffel symbols of textbook GR. Variation with respect to the remaining 16 vierbein variables yields the Einstein field equations.

II. The MacDowell-Mansouri Extension of First order Gravity

The MacDowell-Mansouri idea is to synthesize the vierbein e and the connection ω into a single grand connection A which lives in an O(4,1) internal space. Since there are 4 x 10 = 40 components to such an A, the counting is correct. Two things have to be done to the vierbein e for it to qualify. One is to multiply e by γ_5 in order for it to transform in the adjoint representation of O(4,1). In the formulae to come this will be assumed to have been done. The second modification to e is to multiply it by a parameter with dimension of mass, because that is the appropriate dimensionality for the components of a gauge potential. (e is dimensionless, because its "square" is the dimensionless metric tensor.) So we have

$$A = me + \omega$$

Note that nothing need be done to the γ-matrices themselves, because they live happily in O(4,1) (as long as we specify the square of γ_5 to be -1, contrary to the bj-Drell convention.)

From A one constructs its curvature, or field-strength, F in the standard Yang-Mills way:

$$F = dA + A \wedge A$$

(I will use form language in what follows, but only for the spacetime indices.)

With these preliminaries, the MM Lagrangian density can be written down:

$$\mathcal{L}_{MM} \sim C \; \mathrm{Tr} \; \gamma_5 \, F \wedge F$$

The quantity C is a pure number which normalizes this Lagrangian density appropriately. Note that this Lagrangian looks "almost topological". But it is not, because the γ_5 breaks down the O(4,1) symmetry down to O(3,1). And, after all, there are graviton degrees of freedom lurking deep inside this formalism.

Upon expanding things out, down to the O(3,1) level, one finds three contributions to the Lagrangian density, each of which is even in the parameter m. The first is a cosmological term.

$$\mathcal{L}_{CC} \sim C m^4 \; \mathrm{Tr} \; \gamma_5 \, e \wedge e \wedge e \wedge e$$

The second is the Einstein-Cartan (Hilbert, Palatini) term.

$$\mathcal{L}_{E} \sim C m^2 \; \mathrm{Tr} \; \gamma_5 \, e \wedge e \wedge R$$

Here R is defined as the curvature derived from the O(3,1) connection ω ; it is the Einstein curvature tensor.

$$R = d\omega + \omega \wedge \omega$$

The third term in the Lagrangian, pure topological, is known as the Euler or Gauss-Bonnet term.

$$\mathcal{L}_{GB} \sim C \; \mathrm{Tr} \; \gamma_5 \, R \wedge R$$

The two free parameters in this MM description, C and m , can be fitted to the known coefficients of the cosmological and Einstein terms. Without worrying here about the factors 2, etc., one has

$$G^{-1} \sim M_{pl}^{2} \sim C m^{2} \qquad \text{(Einstein term)}$$

$$G^{-1}\Lambda \sim M_{pl}^{2} H^{2} \sim C m^{4} \qquad \text{(Cosmological term)}$$

Therefore,

$$m \sim H \qquad C \sim \frac{M_{pl}^{2}}{H^{2}} \sim 10^{120} \qquad \left(H^{2} \equiv \frac{\Lambda}{3}\right)$$

Here G is Newton's constant and Λ is the cosmological constant. We also have introduced the constant H as a convenient proxy for Λ; it is the value of the Hubble constant in the limit when dark energy dominates the expansion and H becomes time-independent. (In a future section we will need to also consider the time dependent, dynamical Hubble constant. When that occurs we will distinguish it from this H by writing it as H(t).)

The above equations imply that the scale factor m equals H, indicating that the MM formalism becomes degenerate in the absence of dark energy. Nonvanishing dark energy is a necessary ingredient for the MM description to make sense. While we have not demonstrated it here, the numerical value of m is not only of order H, it actually equals H.

Even more important to note is the extremely large value of C that follows from the above equations. It is another way of expressing the notorious puzzle associated with this large number. In fact, the remainder of this note is devoted to the exploration of the implications of this large number.

Another feature of the MM construction is that deSitter spacetime is a solution of the condition F = 0. This means that, thanks to the presence of the Gauss-Bonnet term, the semiclassical

Wheeler-deWitt wave-function describing the expansion of a piece of dark-energy-dominated spacetime is just a constant independent of time. The usual semiclassical phase factor is absent because the action along the classical path is quadratic in F and vanishes. In some sense, the deSitter limit of spacetime, as described by MM, "goes beyond the quantum theory". If you wish, there are a few more details on this in a talk of mine posted on the arXiv (1008.0033). In these notes (and, in more detail, in an addendum, available upon request), this feature will be used in a much more mundane way--simply to normalize the correct value of the GB coefficient C in a relatively painless way.

There is also an easy way to generalize MM to include CP violation and torsion effects. My way of doing this is described in the aforementioned talk, just beyond Eqn. 24 of arXiv 1008.0033.

III. Gauss-Bonnet Numerology: deSitter Space

It is the large value of the constant C multiplying the Gauss-Bonnet (GB) term which will be the main theme throughout this note. First of all, the GB action, being topological, is typically expressed in terms of integers

$$S_{GB} = 2\pi \int_{t_1}^{t_2} d^4x \, \frac{\partial n}{\partial t}(x,t) = 2\pi \left[N(t_1) - N(t_2) \right]$$

$$N(t) = \int_V d^3x \, n(x,t)$$

This is because the Lagrangian is a pure time derivative, and because the resulting "topological charges" at the quantum level typically take integer values. I here assume this kind of normalization. Then for flat, k = 0, FRW deSitter space it is easy to compute the spatial density n(t), whose integral over a comoving volume gives the integrated charge N. I find it simplest to retreat to the language of textbook metric gravity. Omitting indices and a lot of details, one finds

$$2\pi N(t) = S_{GB}(t) \sim \int^t d^4x \sqrt{g}\,[RR] \sim \frac{M_{pl}^2}{H^2} \int^t dt' \, a^3(t') \left[\left(\frac{\ddot a}{a}\right)\left(\frac{\dot a}{a}\right)^2 \right]$$

$$\sim \frac{M_{pl}^2}{H^2} \int^t dt' \, V(t') H^4 \sim M_{pl}^2 H^2 \int^t dt' \, e^{3Ht'} \sim M_{pl}^2 H \, V(t)$$

Therefore,

$$n = \frac{N(t)}{V(t)} \sim M_{pl}^2 H \sim 10^{-60} M_{pl}^3 \sim (10^{-20} M_{pl})^3 \sim \Lambda_{QCD}^3$$

Derivation of this result is described in more detail in the aforementioned addendum.

Numerically, this (time-independent!) density n is about 60 powers of ten less than the Planck density. Consequently this is of order the QCD density, since the QCD distance scale is about 20 powers of ten larger than the Planck distance scale.

The above relation is essentially what I call the Zeldovich relation (again, see 1008.0033 for a bit more discussion of this). Before the end of this note, this relation will be encountered three more times. Needless to say, I take it as something more significant than a numerical accident.

IV. Gauss-Bonnet Numerology: Cosmology

The "topological charge density" n(t) can be evaluated not only for deSitter space, but for our cosmological history as well. We can do this just by replacing the constant deSitter H in the expression for n(t) by the time-dependent Hubble constant H(t). (Note: it is here we need to be cognizant of the notation I have adopted, which invites confusion if one is not careful. My apologies.) The substitution is as follows:

$$N(t) \sim \frac{M_{pl}^2}{H^2}\int^t dt' \frac{d}{dt'}\, \mathring{a}(t')^3 \sim \frac{M_{pl}^2}{H^2}\, \mathring{a}(t)^3 \sim \frac{M_{pl}^2}{H^2}\, V(t)\, H(t)^3$$

Therefore,

$$n(t) = \frac{N(t)}{V(t)} \sim \frac{M_{pl}^2}{H^2}\, H(t)^3$$

In the early universe, H(t) was much larger than the present value, which is very close to the deSitter H. So it is of interest to determine when the density n(t) was Planckian or greater. Under such circumstances, we may infer that, even though this topological quantity does not influence the Einstein equations, in some real sense the MM formalism must undergo modifications. In other words, under such circumstances the MM formalism is incomplete.

It turns out that the critical time occurs during the radiation dominated epoch, and therefore it is convenient to trade in FRW time t for CMB photon temperature T. The FRW equation, along with the Stefan-Boltzmann expression for temperature in terms of energy density, implies

$$H(t)^2 \sim \frac{1}{M_{pl}^2} \rho_{rad}(t) \sim \frac{T^4}{M_{pl}^2}$$

or

$$H(t) \sim \frac{T^2}{M_{pl}}$$

Therefore the critical temperature T_c satisfies the expression

$$n_c \equiv M_{pl}^3 \sim \frac{M_{pl}^2}{H^2}\left(\frac{T_c^2}{M_{pl}}\right)^3 \sim \frac{T_c^6}{H^2 M_{pl}}$$

and

$$T_c^6 \sim H^2 M_{pl}^4 \sim \Lambda_{QCD}^6$$

Care with numerical factors has been neglected here, but is again covered in the addendum to these notes. But from the above structure, we expect, again because of the presence of the Zeldovich relation, that the critical temperature will be of order Λ_{QCD}. Indeed when the numbers are more carefully crunched, one finds that

$$T_c \approx 50\ MeV \sim \frac{\Lambda_{QCD}}{4}$$

V. Gauss-Bonnet Numerology: Neutron Stars and Other Such Sources

It is also of interest to go back to the present time, and consider the growth of n(t) as one approaches a stationary source of gravity, such as the earth, a star, or a black hole. An easy way to do that is to replace the k=0, cosmological description of dark-energy-dominated deSitter space with a stationary, spherically symmetric description of deSitter space. Then addition of a source term centered at r = 0 can extend this solution to the Schwarzschild-deSitter metric description. Upon moving from large r to small r, the gravitational source term will of course dominate, and at that point the dark-energy contribution can be dropped.

In each step it is not hard to keep track of how the "topological density" $n(t)$ continues to be described. Of course, as one approaches the gravitational source, this density will grow, and again it is of interest to determine when the density becomes Planckian. The answer is best expressed for the case of a source of nuclear matter density, such as a neutron star, a gold nucleus, or for that matter a proton or alpha-particle. We write down the answer first:

$$n(r) = M_{pl}^3 \left(\frac{2.5R}{r}\right)^{9/2}$$

Here R is the radius of the source. Again the reason for this remarkable answer lies in the Zeldovich relation. Again leaving details to the addendum, we sketch the passage through the steps outlined above, leading to the above relationship.

Begin with the FRW metric for deSitter space:

$$ds^2 = dt^2 - a^2(t)\left[d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta\, d\varphi^2\right]$$

Make the replacement $r = a(t)\rho = e^{Ht}\rho$:

$$ds^2 = dt^2 - (dr - v(r)\,dt)^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2$$

$$v(r) = Hr$$

This metric is of the Painleve-Gullstrand form with the special relationship of $v(r)$ to the Hubble parameter H as shown. This form generalizes to the Schwarzschild-deSitter stationary metric in a remarkably simple manner. One simply modifies the formula for $v(r)$ as follows:

$$v^2(r) = \frac{2GM}{r} + H^2 r^2 = \frac{2M}{M_{pl}^2 r} + H^2 r^2$$

The topological density $n(r)$, which is a constant for pure deSitter space, now becomes a function of r, and the formula for it generalizes in a simple way:

$$n(r) = \frac{M_{pl}^2}{H^2}\left[\frac{v^2}{r^2}\frac{dv}{dr}\right]$$

Upon approaching the source, neglect the dark energy contribution and write

$$n(r) \sim \frac{M_{pl}^2}{H^2}\left(\frac{v^3}{2r^3}\right) \sim \frac{M_{pl}^2}{H^2}\left(\frac{M}{M_{pl}^2 r^3}\right)^{3/2} \sim \frac{1}{H^2 M_{pl}}\left(\frac{M}{r^3}\right)^{3/2}$$

Now write the mass M and the radius R of the source as follows:

$$M = M_{proton} A \sim \Lambda_{QCD} A \qquad\qquad R = (1.2f)\, A^{1/3} \sim \Lambda_{QCD}^{-1} A^{1/3}$$

Then

$$n(r) \sim \frac{1}{H^2 M_{pl}}\left(\frac{R}{r}\right)^{9/2} \cdot \left(\frac{\Lambda_{QCD} A}{\Lambda_{QCD}^{-3} A}\right)^{3/2} \sim \frac{\Lambda_{QCD}^6}{H^2 M_{pl}}\left(\frac{R}{r}\right)^{9/2} \sim M_{pl}^3\left(\frac{R}{r}\right)^{9/2}$$

This exhibits the role, as expected, of the Zeldovich relation in arriving at the above conclusion.

I again interpret this as a breakdown of the MM description within regions of nuclear matter density or larger. This may just mean that MM is a description inferior to that of standard metric gravity. I obviously choose not to draw that conclusion, but to assume that there is meaning in the MM formal structure that goes deeper than the phenomenology of the Einstein equations of motion.

VI. Extending the MacDowell Mansouri Extension

The origin of these surprising (at least to me) results is clearly the hugeness of the coefficient C in front of the MM action, leading to the same coefficient in front of the GB term. Why is it so big? Its unnaturalness is itself evidence that the MM description is incomplete. It is more likely an effective action which has emerged from some more fundamental starting point. What might have been "integrated out"?

One straightforward possibility, which is what will be pursued here, is that it is just some extra dimensions that have been integrated out, starting from a generalized MM action of the same form. For example, add a couple of extra compactified dimensions, introduce an O(6,1) connection A and associated field strength F , and write

$$\mathcal{L} \sim \mathrm{Tr}\, \gamma_7\, F \wedge F \wedge F$$

Imagine that, in the pair of extra dimensions, F is of Planckian scale, and that those two extra dimensions are large compared to the Planck distance scale, of size ℓ. Then, schematically, one might imagine a scenario where the effects of the two extra dimensions easily "integrate out", leaving an effective O(4,1) MM theory with a coefficient C of order $M_{pl}^2 \ell^2 >> 1$.

To get a factor 10^{120} this way with only two extra dimensions is awkward. Trial and error, along with political correctness, rapidly converges on the most attractive choice, namely six extra "large" dimensions. The internal symmetry group is now O(10,1) (or perhaps a variant like O(8,3) or O(6,5)), and the Lagrangian density is

$$\mathcal{L} \sim \mathrm{Tr}\, \gamma_{11}\, F \wedge F \wedge F \wedge F \wedge F$$

After integrating out the six extra dimensions in the same style as our example above, we will obtain the coefficient C as

$$C \sim \left[\int d^2 z\, F\right]^3 \sim \left[M_{pl}^2 \ell^2\right]^3 \sim 10^{120}$$

This implies, as shown, that the size ℓ of the extra compact dimensions is of the QCD scale.

This result provides something of an a posteriori reason for understanding why the previous results on the limits of the applicability of the MM description should apply. For distance scales larger than the QCD scale the description has the six extra dimensions fully "integrated out". It is utterly unreasonable that, as the distance scale is taken to be smaller or equal to the QCD scale, the impact of those extra dimensions does not affect the nature of the description. In particular, the "topological charge" N we have been considering may enter all ten dimensions, which of course will dilute its (ten-dimensional) density far below the Planck scale.

The real challenge for this viewpoint is to ensure that the opening up of these extra dimensions at such a large scale does not ruin well-established, precise, standard model phenomenology.

There should for example be no low-scale Kaluza-Klein modes interfering with existing experimental constraints. Fermionic degrees of freedom should lie on boundaries of the internal space, for the most part away from the bulk. There may be guidance here from the description of topological insulators, and condensed-matter experience with the Hall effect, etc. But independently of how high the risk factor is, I much prefer worrying about this problem instead of pondering the description of 10^{500} member universes within the string-theory landscape.

VII. Some Comments

There are some clear echoes of string theory in this scenario, despite this extended MM action being only applicable as an effective field theory taken at the QCD scale and having nothing directly to do with supersymmetry. Therefore this extra six-dimensional manifold need not be Calabi-Yau. Nevertheless, it might inherit common features if it turns out that this MM-extension effective theory is the infrared descendant of an ultraviolet string-theory starting point.

With this in mind, I sought out a guru, Shamit Kachru. He gave me splendid advice on how to learn something of the Calabi-Yau trade. It is to read Yau's popular book "The Shape of Inner Space". It is a superb survey of string theory technology, within a pictorial and intuitive setting that even I can follow reasonably well. I highly recommend it. I am still not competent in stringy ways of thinking. But, thanks to Yau, I am no longer intimidated by the prospect of dealing with these extra dimensions in the style of Kahler, Calabi-Yau, et. al.

However, the general problem remains intimidating, at least to me. The MM potential A now has 550 components, and a single field strength F has 2475 components. Only 40 of the A's (and a mere 60 of the F's) are directly relevant to Einstein gravity. Clearly one wants to exploit the remaining elements as candidate standard-model degrees of freedom. There is an easy O(6) subgroup to exploit, enough to accommodate the SU(3) color group. I regard identifying the gluon degrees of freedom within the MM fields as the first order of business. This is not totally trivial, because the "topological" texture of this extended MM action conflicts a bit with the structure of the desired Maxwell kinetic term for the gluons.

Of course one does not want to stop with gluons alone. But it is not at all clear that the electroweak sector, plus three generations of fermions, plus an appropriate Higgs sector, can all coexist with QCD degrees of freedom inside this structure. I think it Is most prudent to build things up a step at a time---starting, as indicated above, with the QCD sector. To me this option is also highlighted by the ubiquity of the Zeldovich relation, indicating perhaps that QCD has an exceptionally important role to play, especially given that this scale is the characteristic scale for the MM-extension description itself.

Since the scale at which the extra dimensions open up is the same as the scale of the quark and lepton masses, it is attractive to try to link the origins of mass and CKM/MNS mixings to the presence of these large extra dimensions. This is what has presently attracted my attention the most. I have opted to study the "rotating mass matrix" scenario of Hong-Mo Chan (arXiv 1103.5615) as the description most easily adaptable in principle to the problem at hand. This is a high-risk judgment call, but for better or worse that is where I am at present.

In this regard, it is interesting to me that on a logarithmic scale the centroid of the mass spectrum of quarks and leptons lies near Λ_{QCD}. For example, the geometric mean of the top quark mass and the electron mass is of order 300 MeV . The usual Higgs picture does not give much of a clue as to why this should be true, while this extra-dimensional viewpoint might in the long run provide a rationale.

Another frontier which is very interesting, but which I have set aside for now, is to try to understand the nature of the "topological density" $n(t)$ which was central to the whole line of argument in this note. Can one identify it with some quantity defined within the standard model? Especially because of the numerology, it is tempting to compare its role with the role of Skyrmions in the QCD chiral effective action. In the Skyrmion analogy, the topological charge density is baryon number. The density $n(t)$ cannot be that. But maybe there are some lessons to learn in making the comparison. And I feel that somehow the N and the Θ_{QCD} parameters of strong CP-violation may somehow be a part of the story line.

In any case, there remain a lot of things to do.