Partial Fractions (PF)

we continue our discussion with (PF)

So far

$$\int \frac{8x-1}{(2x+1)(3x-1)}\,dx \qquad \frac{A}{2x+1} + \frac{B}{3x-1}$$

Same for 3 linear factors

Repeated factors

$$\frac{4x^2+9x+4}{(x+1)^2(2x+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{2x+1}$$

Now we consider quadratic factors

Ex 1 $\int \frac{3x^2+x+1}{x(x^2+1)}\,dx$ try $\frac{A}{x} + \frac{B}{x^2+1}$

but we need 3 constants

order of denom. is 3

Consider

$$\frac{A}{x} + \frac{B}{x^2} = \frac{Ax+B}{x^2} \quad \begin{matrix} \leftarrow \text{linear} \\ \leftarrow \text{quadratic} \end{matrix}$$

so

$$\frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{3x^2+x+1}{x(x^2+1)}$$

$$A(x^2+1) + (Bx+C)x = 3x^2+x+1$$

x^2) $A + B = 3 \Rightarrow B = 2$

x) $+C = 1$

1) $A = 1$

$$\int \frac{1}{x} + \frac{2x+1}{x^2+1}\,dx$$

$$\int \left(\frac{1}{x} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}\right) dx = \ln|x| + \ln|x^2+1| + \tan^{-1}x + C$$

$u = x^2+1$

Ex 2 $\displaystyle\int \frac{6x^3+x^2+18x+4}{(x^2+1)(x^2+4)}\,dx$

$$\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} = \frac{6x^3+x^2+18x+4}{(x^2+1)(x^2+4)}$$

$$(Ax+B)(x^2+4) + (Cx+D)(x^2+1) = 6x^3+x^2+18x+4$$

$x^3)$ $A+C = 6$

$x^2)$ $B+D = 1$

$x)$ $4A+C = 18$

$1)$ $4B+D = 4$

$C = 6-A$

$4A+6-A=18$

$3A = 12 \quad A = 4$

$\Rightarrow C = 2$

$B+D=1 \rightarrow D = 1-B$

$4B+1-B = 4 \Rightarrow B = 1$

$D = 0$

so $\displaystyle\int \frac{4x+1}{x^2+1} + \frac{2x}{x^2+4}\,dx$

$= 2\ln|x^2+1| + \tan^{-1}x + \ln|x^2+4| + C$

Ex <u>Repeated Quadratic factors</u>

Ex 3 $\displaystyle\int \frac{2x^3+6x+1}{(x^2+1)^2}\,dx$

$$\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}=\frac{2x^3+6x+1}{(x^2+1)^2}$$

$$(Ax+B)(x^2+1)+(Cx+D)=2x^3+6x+1$$

$x^3)\quad A = 2$

$x^2)\quad B = 0$

$x)\quad A+C = 6$

$1)\quad B+D = 1$

$A=2 \quad B=0$

$C=4 \quad D=1$

$$\int \frac{2x}{x^2+1}+\frac{4x+1}{(x^2+1)^2}\,dx = \int \underbrace{\frac{2x}{x^2+1}+\frac{4x}{(x^2+1)^2}}_{u=x^2+1}+\frac{1}{(x^2+1)^2}\,dx$$

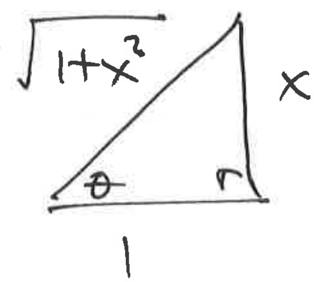
$$=\ln|x^2+1|-\frac{2}{x^2+1}$$

For $\int \frac{1}{(1+x^2)^2}\,dx$

trig sub $x = \tan\theta$ $\quad dx = \sec^2\theta\,d\theta$

$$\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} .$$ (+C later)



$$= \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}$$

$$= \frac{1}{2}\tan^{-1}x + \frac{x}{2(1+x^2)}$$

$\sin\theta = \frac{x}{\sqrt{1+x^2}}$ $\quad \cos\theta = \frac{1}{\sqrt{1+x^2}}$

Ans

$$= \ln|x^2+1| - \frac{2}{x^2+1} + \frac{1}{2}\tan^{-1}x + \frac{x}{2(1+x^2)} + C$$

Definite Long!

Ex 4 $\displaystyle\int \frac{x^4+9x^2+16}{x^2(x^2+4)^2}\,dx$

$$\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+4}+\frac{Ex+E}{(x^2+4)^2}=\frac{x^4+9x^2+16}{x^2(x^2+4)^2}$$

$$Ax(x^2+4)^2+B(x^2+4)^2+(Cx+D)x^2(x^2+4)+(Ex+F)x^2=$$

Gets to be a long example

Note: Since this is true for all x

then certainly for $x=0$

$$0+16B+0+0=0+0+16$$

$$\Rightarrow B=1$$

You could pick other values

$x=\pm1,\quad x=\pm2$

then you would have 4 eq^n's for A, C, D, E, F

would need one eq^n