Math 3331 - ODE's

1st order

$$\frac{dy}{dx} = F(x,y)$$

Techniques

(1) separable

$$\frac{dy}{dx} = f(x)\,g(y)$$

separate & integrate $\quad \displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx + c$

(2) Linear

SF $\quad \dfrac{dy}{dx} + p(x)\,y = q(x)$

integrating factor $\quad \mu = e^{\int p(x)\,dx}$

mult. by μ DE becomes $\quad \dfrac{d}{dx}(\mu y) = \mu q \leftarrow$ separate & integrate

Ex1 $\quad \frac{dy}{dx} + \frac{2x}{x^2+1} y = 12x \qquad y(0) = 1$

already standard form.

$$p(x) = \frac{2x}{x^2+1} \qquad \mu = e^{\int p\,dx} = e^{\int \frac{2x\,dx}{x^2+1}} = e^{\ln(x^2+1)}$$

$$= x^2+1$$

so mult by μ gives

$$\frac{d}{dx}(x^2+1)y = 12x(x^2+1)$$

$$\int d((x^2+1)y) = \int 12x^3 + 12x\,dx$$

$$(x^2+1)y = \frac{12x^4}{4} + \frac{12x^2}{2} + c$$

$$(x^2+1)y = 3x^4 + 6x^2 + c$$

$y(0) = 1 \Rightarrow \quad 1 = c$

$$y = \frac{3x^4 + 6x^2 + 1}{x^2+1}$$

Note:

if you can't integrate

$\mu g(x)$

leave it as

$\int \mu g(x)\,dx$

(3) Bernoulli Eq^n

$$\frac{dy}{dx} + p(x)\, y = q(x)\, y^n \qquad n \neq 0 \ - \text{linear}$$

$$n \neq 1 \ - \text{sep}$$

n is a #

Ex 2 $\quad x \dfrac{dy}{dx} + y = x^2 y^2$

SF $\quad \dfrac{dy}{dx} + \dfrac{y}{x} = x y^2$

Divide by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x} \cdot \frac{y}{y^2} = x$$

let $u = \dfrac{1}{y}$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \quad \text{this piece w/o -ve}$$

$$-\frac{du}{dx} + \frac{u}{x} = x \qquad \text{linear}$$

$$\frac{du}{dx} - \frac{u}{x} = -x \qquad \text{SF linear,}$$

$$\mu = e^{-\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$$

mult by $\frac{1}{x}$

$$\frac{d}{dx}\left(\frac{1}{x} \cdot u\right) = -\frac{x}{x} = -1$$

Sep $\quad d\left(\frac{1}{x} \cdot u\right) = -dx$

$\int \quad \frac{1}{x} u = -x + c$

back sub $\quad u = \frac{1}{y}$ so $\quad \frac{1}{xy} = -x + c$

a $\quad y = \dfrac{1}{x(c-x)}$

<u>Ex 2</u> $\quad xy^2 \frac{dy}{dx} = y^3 - x^3 \quad y(1) = 2$

$$\frac{dy}{dx} = \frac{y^3 - x^3}{xy^2} = \frac{y^3}{xy^2} - \frac{x^3}{xy^2}$$

$$= \frac{y}{x} - \frac{x^2}{y^2}$$

or $\frac{dy}{dx} - \frac{y}{x} = -x^2 y^{-2}$ SF Bern

move this y term

mult. by y^2

$$y^2 \frac{dy}{dx} - \frac{y^3}{x} = -x^2 ,$$

let $u = y^3$ $\frac{du}{dx} = 3y^2 \frac{dy}{dx}$

$$\frac{1}{3}\frac{du}{dx} - \frac{u}{x} = -x^2 \quad \text{(linear)}$$

$$\frac{du}{dx} - \frac{3u}{x} = -3x^2 \quad \text{(SF linear)}$$

$$\mu = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = \frac{1}{x^3}$$

$$\frac{d}{dx}\left(\frac{1}{x^3}u\right) = \frac{-3x^2}{x^3} = -\frac{3}{x} \leftarrow \text{Sep} \;\&\; \int$$

$$\frac{1}{x^3}u = -3\ln|x| + c$$

$y(1) = 2$

$$\Rightarrow \frac{2^3}{1^3} = -3\ln|1| + c \Rightarrow c = 8$$

$$\frac{y^3}{x^3} = -3\ln|x| + c$$

$$\boxed{\text{sol}^n \quad \frac{y^3}{x^3} = 8 - 3\ln|x|}$$