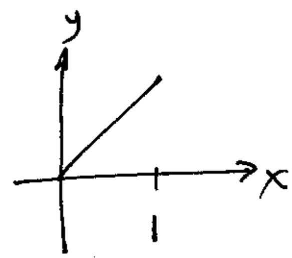
Math 2471 — Calc 3

Line integrals over VF

Consider

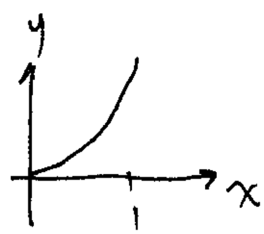
$$\int_C y\,dx + x\,dy$$ when C is along $y = x$ $[0,1]$



so $dy = dx$

$$\int_0^1 x\,dx + x\,dx = \int_0^1 2x\,dx = x^2\Big|_0^1 = 1$$

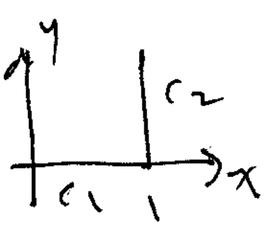
C: $y = x^2$ $[0,1]$



so $dy = 2x\,dx$

$$\int_0^1 x^2\,dx + x\cdot 2x\,dx = \int_0^1 3x^2\,dx = x^3\Big|_0^1 = 1$$ same

C:



c_1 $y = 0$ $dy = 0$ $\qquad \int_{c_1} \to = 0$

c_2 $x = 1$ $y: 0 \to 1$

$dx = 0$

$$\int_0^1 1\,dy = y\Big|_0^1 = 1$$ same — maybe there is something special about VF

The VF is conservative

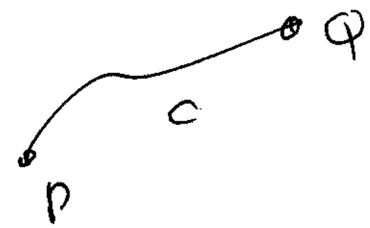
$\int P\,dx + Q\,dy$ when $P_y = Q_x$

so f exists $P = f_x$ & $Q = f_y$

so $\int_c f_x\,dx + f_y\,dy = \int_c df = f\Big|_P^Q$

this is the fundamental Th^m of line $\int$'s

if $\int_c P\,dx + Q\,dy + R\,dz$



when $\bar{\nabla}\times\bar{F} = \bar{\nabla}\times\langle P, Q, R\rangle = \bar{0}$ then VF is cons

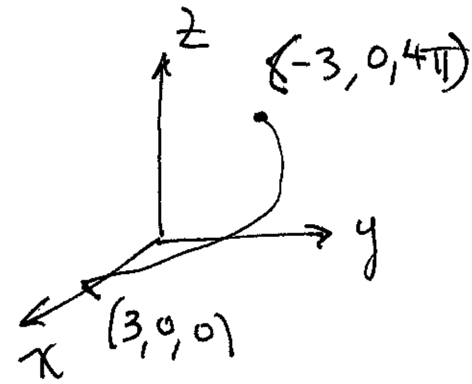
$= \int_c df = f\Big|_P^Q$

Q $\bar{E}$

Evaluate $\int_c (2xz + y)\,dx + x\,dy + (x^2 - 3z^2)\,dz$

where c: $\vec{r} = \langle 3\cos t, 3\sin t, 4t \rangle$

$0 \le t \le \pi$



Is the VF conservative?

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz+y & x & x^2-3z^2 \end{vmatrix}$$

$= \langle 0, -(2x-2x), 1-1 \rangle = \bar{0}$ so yes

$f_x = 2xz+y \Rightarrow f = x^2 z + xy + A(y,z)$

$f_y = x \Rightarrow f = xy + B(x,z)$

$f_z = x^2 - 3z^2 \Rightarrow f = x^2 z - z^3 + C(x,y)$

$f = x^2 z + xy - z^3 + c$ ← don't need this c here

so $\int_C (2xz+y)\,dx + x\,dy + (x^2-3z^2)\,dz$

$$= x^2 z + xy - z^3 \Big|_{(3,0,0)}^{(-3,0,4\pi)} = \left(9 \cdot 4\pi - (4\pi)^3\right) - (0 = 36\pi - 4^3\pi^3$$

so if we evaluate

$$\oint_C P\,dx + Q\,dy$$ where C is a simple closed path.

and if $\bar{F} = \langle P, Q \rangle$ is conservative then

$$\oint_C P\,dx + Q\,dy = 0$$

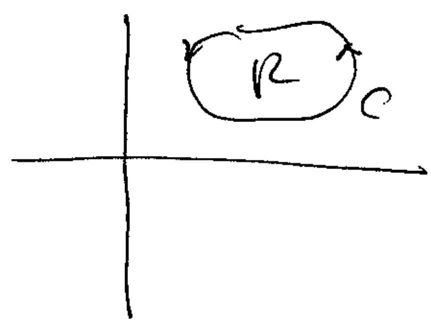
- closed simple path -
curve doesn't cross itself - no
holes so ∞ or ← No

But suppose $\bar{F}$ is not conservative -
Can we do maybe a simpler calculation

<u>Green's Thm</u>

if $\oint_C P\,dx + Q\,dy$ when C is a simple closed curve, oriented CCW
& P, Q have cont's derivatives then

$$\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$$



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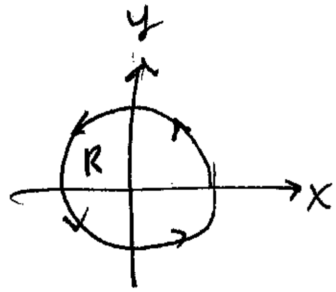
$\vec{F} = \langle 0, x^2+y^2 \rangle \qquad R: \{(x,y) \mid x^2+y^2 \le 1\}$



so $\oint_C (x^2+y^2)\,dy$

$x = \cos t,\ y = \sin t$

$dy = \cos t\,dt \qquad x^2+y^2 = 1$

$$\int_0^{2\pi} 1 \cdot \cos t\,dt = \sin t \Big|_0^{2\pi} = 0$$

Now $Q_x - P_y = 2x - 0$

$$\iint_R 2x\,dA = \int_0^{2\pi}\int_0^1 2r\cos\theta\; r\,dr\,d\theta$$

$\xrightarrow{\text{polar}}$

the ~~review G.T. pg 1098~~
Hand out.

$$= \int_0^{2\pi} \frac{2r^3}{3}\Big|_0^1 \cos\theta\,d\theta = \frac{2}{3}\int_0^{2\pi} \cos\theta\,d\theta$$

$$= \frac{2}{3}\sin\theta\Big|_0^{2\pi} = 0 \quad \text{same}$$

Note: $\oint_C (x^2+y^2)\,dy = 0$ ~~does~~ doesn't imply VF is conser.