Math 3331 - ODEs

Consider $y'' - y = 4xe^{x}$

1st comp $y'' - y = 0$

CE $m^2 - 1 = 0$ $m = \pm 1$ $y_c = c_1 e^{x} + c_2 e^{-x}$

Red. order $y_p = e^{x} u$

M of UD $y_p = Ax^2e^{x} + Bxe^{x}$

$y_p' = Ax^2e^{x} + 2Axe^{x} + Bxe^{x} + Be^{x}$

$y_p'' = Ax^2e^{x} + 2Axe^{x} + 2Axe^{x} + 2Ae^{x} + Bxe^{x} + 2Be^{x}$

sub $y_p'' - y = 4xe^{x}$

$$\cancel{Ax^2e^{x}} + 4Axe^{x} + 2Ae^{x} + \cancel{Bxe^{x}} + 2Be^{x} = 4xe^{x}$$
$$-\cancel{Ax^2e^{x}} \qquad -\cancel{Bxe^{x}}$$

$4A = 4$ $2A + 2B = 0$

$A = 1$ $B = -1$ $y_p = x^2e^{x} - xe^{x}$

New Technique Variation of parameters

$$y = ue^{x} + ve^{-x}$$

$$y' = u'e^{x} + ue^{x} + v'e^{-x} - ve^{-x}$$

if we continue we'll have 1 eq^n for u & v

we have some flexibility here.

choose $u'e^{x} + v'e^{-x} = 0$

so $y' = ue^{x} - ve^{-x}$

$$y'' = u'e^{x} + ue^{x} - v'e^{-x} + ve^{-x}$$

sub $y'' - y = 4xe^{x}$

$$u'e^{x} + \cancel{ue^{x}} - v'e^{-x} + \cancel{ve^{-x}} = 4xe^{x}$$
$$\quad - \cancel{ue^{x}} \qquad - \cancel{ve^{-x}}$$

So (1) $u'e^{x} + v'e^{-x} = 0$

(2) $u'e^{x} - v'e^{-x} = 4xe^{x}$

$2u'e^{x} = 4xe^{x}$

$u' = 2x$

$2v'e^{-x} = -4xe^{x} \Rightarrow v' = -2xe^{+2x}$

we now integrate

$$u = x^2 + c_1 \quad v = -\frac{2x\cancel{+}1}{2} e^{+2x} + c_2$$

$$y = u e^{x} + v e^{-x}$$

$$= (x^2 + c_1) e^{x} + \left\{ \cancel{(}(-x + \tfrac{1}{2}) e^{+2x} + c_2 \right\} e^{-x}$$

$$= x^2 e^{x} - x e^{x} + \frac{1}{2} e^{x} \quad + c_1 e^{x} + c_2 e^{-x}$$

$$y = \underbrace{x^2 e^{x} - x e^{x}}_{y_p} + \underbrace{c_1 e^{x} + c_2 e^{-x}}_{y_c}$$

Ex2 $\quad y'' + y = \sec x$

$y_c \quad y'' + y = 0 \quad m^2 + 1 = 0 \quad m = \pm i$

$$y_c = c_1 \cos x + c_2 \sin x$$

VP $\quad y = u\cos x + v\sin x$

$$y' = u'\cos x - u\sin x + v'\sin x + v\cos x$$

choose $\quad u'\cos x + v'\sin x = 0$

so $\quad y' = -u\sin x + v\cos x$

$$y'' = -u'\sin x - u\cos x + v'\cos x - v\sin x$$

sub $\quad y'' + y = \sec x$

$$-u'\sin x - \cancel{u\cos x} + v'\cos x - \cancel{v\sin x} + \cancel{u\cos x} + \cancel{v\sin x} = \frac{1}{\cos x}$$

(1) $\quad u'\cos x + v'\sin x = 0$

$$-u'\sin x + v'\cos x = \frac{1}{\cos x}$$

$$u' = -\frac{v'\sin x}{\cos x} \qquad -\left(-\frac{v'\sin x}{\cos x}\right)\sin x + v'\cos x = \frac{1}{\cos x}$$

$$v'\left(\frac{\sin^2 x}{\cos x} + \cos x\right) = \frac{1}{\cos x} \Rightarrow v' = 1$$

So $u' = -\dfrac{\sin x}{\cos x}$

integrate $u = \ln|\cos x|$ $\quad v = x$

$$y_p = \cos x \ln|\cos x| + x \sin x$$

$$y_g = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$$

Suppose $y(0) = 0$ $\quad y'(0) = 1$

$$y(0) = c_1 + 0 + 1 \ln(1) + \; = 0 \Rightarrow c_1 = 0$$

$$y' = c_2 \cos x + \cancel{\cos x}\,\frac{(-\sin x)}{\cancel{\cos x}} - \sin x \ln|\cos x|$$

$$+ \sin x - x \cos x$$

$$y'(0) = c_2 = 1$$

$$y = \sin x + \cos x \ln|\cos x| + x \sin x.$$