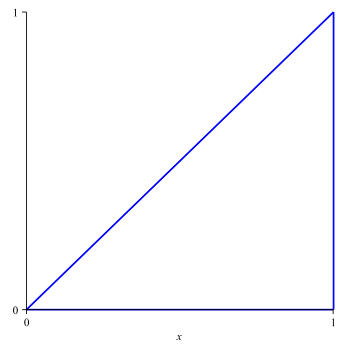
Calculus 3 - Cylindrical and Spherical Coordinates

In Calculus 2 we introduced polar coordinates

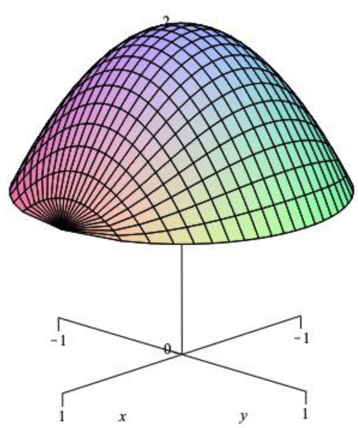
$$x = r\cos\theta, \quad y = r\sin\theta, \tag{1}$$

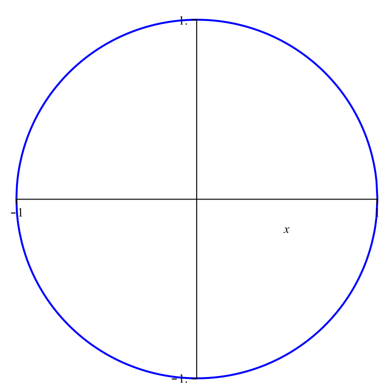
and

$$x^2 + y^2 = r^2, \quad \tan\theta = \frac{y}{x}. \tag{2}$$



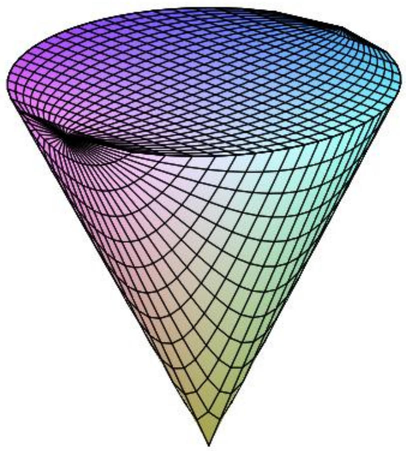
Last week we saw that some double integrals would be easier in this coordinate system. For example, the volume under the paraboloid $z = 2 - x^2 - y^2$ and inside the cylinder $x^2 + y^2 = 1$, for $z \geq 0$

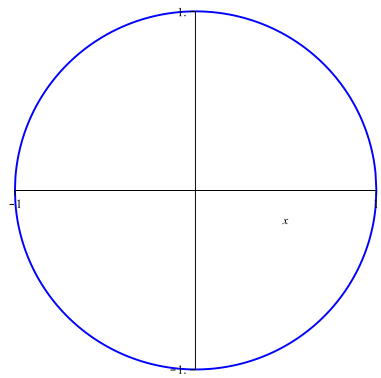




$$V = \int_0^{2\pi} \int_0^1 (2 - r^2) r \, dr \, d\theta \tag{3}$$

Last week we introduced triple integrals. For example, set up the triple integral for the volume bound by the cones $z = \sqrt{x^2+y^2}$ and the plane $z = 1$





In Cartesian coordinates, the setup would be

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} F(x,y,z)dz\,dy\,dx. \tag{4}$$

Maybe there are "polar" coordinates in 3D would make triple integrals easier. Here we introduce two types of polar coordinates.

Cylindrical Polar Coordinates

These are similar to the usual polar coordinates that are in 2D but we simply add z to extend to 3D so

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \tag{5}$$

Let us consider a number of quadratic surfaces.

Sphere

$$x^2 + y^2 + z^2 = a^2 \quad \Rightarrow \quad r^2 + z^2 = a^2 \tag{6}$$

Hyperboloid of 1 Sheet

$$x^2 + y^2 - z^2 = 1 \quad \Rightarrow \quad r^2 - z^2 = 1 \tag{7}$$

Hyperboloid of 2 Sheets

$$-x^2 - y^2 + z^2 = 1 \quad \Rightarrow \quad -r^2 + z^2 = 1 \tag{8}$$

Cone

$$x^2 + y^2 = z^2 \quad \Rightarrow \quad r^2 = z^2 \text{ or } z = \pm r \tag{9}$$

Paraboloid

$$z = x^2 + y^2 \quad \Rightarrow \quad z = r^2 \tag{10}$$

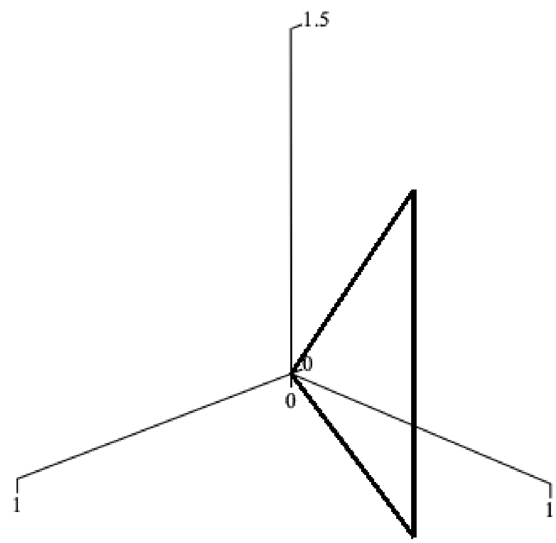
Notice that each of these equations are independent of θ so draw the picture in the rz plane and rotate $0 \to 2\pi$.

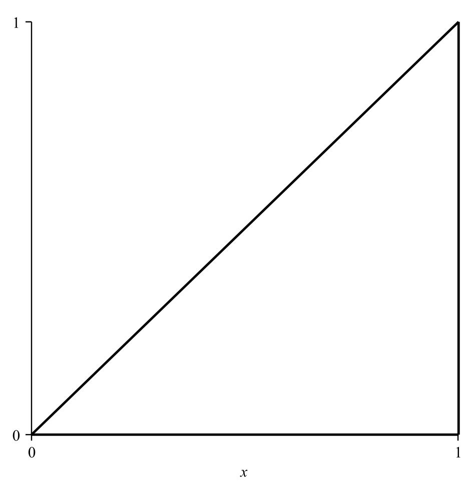
Plane

$$x + y + z = 1 \quad \Rightarrow \quad r\cos\theta + r\sin\theta + z = 1 \tag{11}$$

This one is better in Cartesian!

Spherical Polar Coordinates





We see that

$$\sin\phi = \frac{r}{\rho}, \quad \cos\phi = \frac{z}{\rho}, \tag{12}$$

or

$$r = \rho \sin\phi \quad z = \rho\cos\phi \tag{13}$$

and since

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \tag{14}$$

then

$$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\phi. \tag{15}$$

Now, we see that

$$x^2 + y^2 = \rho^2\sin^2\phi \quad so \quad \sqrt{x^2+y^2} = \rho\sin\phi \tag{16}$$

So from (15) and (16)

$$\tan\theta = \frac{y}{x}, \quad \tan\phi = \frac{\sqrt{x^2+y^2}}{z}, \quad x^2+y^2+z^2 = \rho^2. \tag{17}$$

Some Surfaces

Sphere

$$x^2+y^2+z^2 = a^2 \quad \Rightarrow \quad \rho^2 = a^2 \;\; or \;\; \rho = a \tag{18}$$

Hyperboloid of 1 Sheet

$$x^2+y^2-z^2 = 1, \quad \Rightarrow \quad \rho^2\sin^2\phi - \rho^2\cos^2\phi = 1 \tag{19}$$

Hyperboloid of 2 Sheets

$$-x^2-y^2+z^2 = 1 \quad \Rightarrow \quad -\rho^2\sin^2\phi + \rho^2\cos^2\phi = 1 \tag{20}$$

Cone

$$x^2 + y^2 = z^2 \quad \Rightarrow \quad \rho^2 \sin^2 \phi = \rho^2 \cos^2 \phi \;\; \text{or} \;\; \tan \phi = 1 \;\; \textit{so} \;\; \phi = \frac{\pi}{4} \tag{21}$$

Paraboloid

$$z = x^2 + y^2 \quad \Rightarrow \quad \rho \cos \phi = \rho^2 \sin^2 \phi \tag{22}$$

so we see that in cylindrical coordinates the

hyperboloid of 1 and 2 sheets, and the *paraboloid*

becomes easier, whereas in spherical polar coordinates the

sphere and the *cone*

becomes easier.

Triple Integrals

We consider triple integrals

$$V = \iiint\limits_V F(x, y, z) dV \tag{23}$$

in each of these two coordinate systems. We first consider cylindrical and then spherical.

Cylindrical Coordinates

This integral has three main parts:

1. the integrand
2. dV
3. limits

1. The integrand

For this part, we simplify substitute in

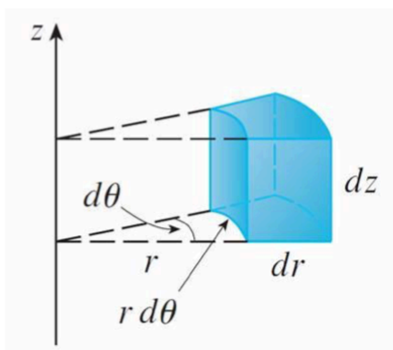
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \tag{24}$$

into $F(x, y, z)$ and simplify. So, in general,

$$\iiint\limits_V F(x, y, z) dA = \iiint\limits_V F(r\cos\theta, r\sin\theta, z) dV. \tag{25}$$

2. dV

In cartesian coordinates, this is $dV = dxdydz$. In cylindrical coords



$$dV = ds\, dr\, dz = rdr\, d\theta\, dz \tag{26}$$

3. Limits of Integration

These ultimately come from the picture of the volume itself. So

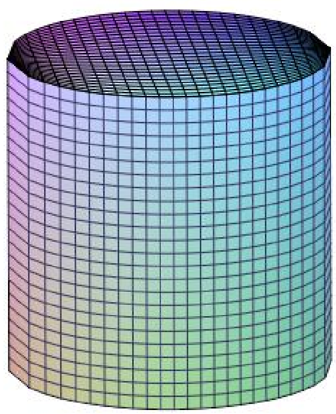
$$\int_\alpha^\beta \int_{r_i(\theta)}^{r_o(\theta)} \int_{f_1(r,\theta)}^{f_2(r,\theta)} F(r\cos\theta, r\sin\theta, z) rdz\, dr\, d\theta \tag{27}$$

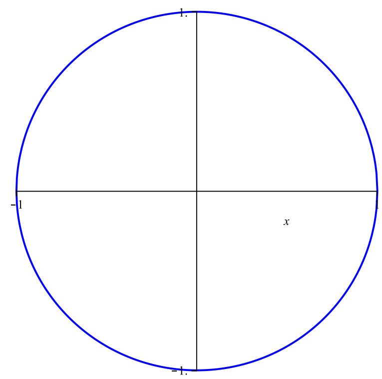
where $r = r_i(\theta)$ is the inner curve and $r = r_o(\theta)$ is the outer curve.

Example 1. Set up the triple integral for the volume bound by $z = 0, z = 1$, and inside the cylinder $x^2 + y^2 = 4$ and evaluate

$$\iiint\limits_V z dV \tag{28}$$

Soln:



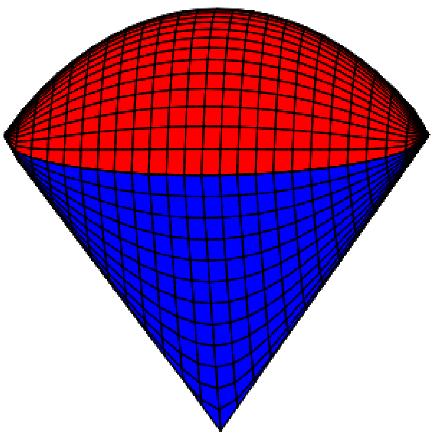


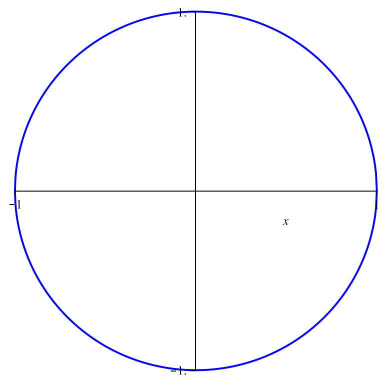
$$\begin{aligned}
\int_0^{2\pi} \int_0^2 \int_0^1 z\, r\, dz\, drd\theta &= \int_0^{2\pi} \int_0^2 r\, \frac{1}{2} z^2 \Big|_0^1 dr\, d\theta \\
&= \frac{1}{2} \int_0^{2\pi} rdr\, d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \frac{1}{2} r^2 \Big|_0^2 d\theta \\
&= \frac{4}{4} \int_0^{2\pi} d\theta \\
&= \theta \Big|_0^{2\pi} = 2\pi.
\end{aligned} \tag{29}$$

Example 2. Set up the triple integral for the volume bound by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 2$ and evaluate

$$\iiint\limits_V z dV \tag{30}$$

Soln:





We first convert each surface to cylindrical polar coords. They become $z = r$ and $r^2+z^2 = 2$. Eliminating z gives $2r^2 = 2$ or $r = 1$

$$\begin{aligned}\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} z\,r\,dz\,drd\theta &= \int_0^{2\pi}\int_0^1 r\,\frac{1}{2}z^2\Big|_r^{\sqrt{2-r^2}} dr\,d\theta\\ &= \frac{1}{2}\int_0^{2\pi}\int_0^1 r\,\left(2-r^2-r^2\right) dr\,d\theta\\ &= \frac{1}{2}\int_0^{2\pi}\int_0^1 \left(2r-2r^3\right) dr\,d\theta\\ &= \frac{1}{2}\int_0^{2\pi} r^2 - \frac{1}{2}r^4\Big|_0^1 d\theta\\ &= \frac{1}{4}\,\theta\Big|_0^{2\pi} = \frac{\pi}{2}.\end{aligned} \tag{31}$$