Math 2371 Calc III
Sample Test 3 - Solns

1. Is the following vector field conservative?

$$\vec{F} =< 2xy, x^2 + z^2, 2yz > .$$

Soln. Since $\nabla \times \vec{F} = 0$ then yes, the vector field is conservative. Thus f exists such that $\vec{F} = \vec{\nabla} f$ so

$$\begin{aligned}
f_x &= 2xy & &\Rightarrow & f &= x^2y + A(y,z) \\
f_y &= x^2 + z^2 & &\Rightarrow & f &= x^2y + yz^2 + B(x,z) \\
f_z &= 2yz & &\Rightarrow & f &= yz^2 + C(x,y)
\end{aligned}$$

Therefore we see that

$$f = x^2y + yz^2 + c$$

and

$$\int_C 2xy\,dx + (x^2 + z^2)\,dy + 2yz\,dz = x^2y + yz^2\Big|_{(1,1,1)}^{(2,3,4)} = 58$$

2. Evaluate the following line integral $\displaystyle\int_c y\,ds$ where c is ccw direction around the circle $x^2 + y^2 = 1$ from $(1,0)$ to $(-1,0)$.

Soln. Here we parameterize the curve by $x = \cos t, \quad y = \sin t$ so $t = 0$ to $t = \pi$. Also, $\dfrac{dx}{dt} = \sin t$ and $\dfrac{dy}{dt} = -\cos t$ so $ds = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}dt = \sqrt{\sin^2 t + \cos^2 t}dt = dt$. Thus, the line integral becomes

$$\int_0^\pi \sin t dt = 2$$

3. Evaluate the following line integral $\int_c 2y\,dx + x\,dy$ where c is clockwise direction around the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

Soln. Here we have three separate curves which we denote by C_1, C_2 and C_3.

C_1 : Here $y = 0, dy = 0$ so $\displaystyle\int_{C_1} 0 = 0$

C_2 : Here $x = y, dx = dy$ so $\displaystyle\int_0^1 3x\,dx = 3/2$

C_3 : Here $x = 1, dx = 0$ so $\displaystyle\int_1^0 dy = -1$

Thus $\int\limits_{c} 2y\,dx + x\,dy = 0 + 3/2 - 1 = 1/2.$

4. Green's Theorem is

$$\int\limits_{C} P\,dx + Q\,dy = \iint\limits_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Verify Green's Theorem where $\vec{F} =< y^2, x^2 + 2xy >$ where R is the region bound by the curves $y = x^2$, $y = 1$ and $x = 0$ in $Q1$.

Soln. Again, we have three separate curves which we denote by C_1, C_2 and C_3.

C_1 : Here $y = x^2, dy = 2x\,dx$ so $\displaystyle\int_0^1 x^4 dx + (x^2 + 2x^3) 2x\,dx = 3/2$

C_2 : Here $y = 1, dy = 0$ so $\displaystyle\int_1^0 dx = -1$

C_3 : Here $x = 0, dx = 0$ so $\displaystyle\int_{C_3} 0 = 0$

Thus $\int_c y^2 dx + (x^2 + 2xy) dy = 3/2 - 1 + 0 = 1/2.$

Since $P = y^2$ and $Q = x^2 + 2xy$ then $Q_x - P_y = 2x + 2y - 2y = 2x$ so

$$\iint\limits_{R} (Q_x - P_y)\,dA = \int_0^1 \int_{x^2}^1 2x dy\,dx = 1/2$$

5. Evaluate $\iint\limits_{S} z\,dS$ where S is the surface of the paraboloid $z = 1 - x^2 - y^2, z \geq 0$.

Soln. Since $z = 1 - x^2 - y^2$ then $dS = \sqrt{1 + z_x^2 + z_y^2}\,dA = \sqrt{1 + 4x^2 + 4y^2} dA$ and so far we have $\iint\limits_{R} (1 - x^2 - y^2) \sqrt{1 + z_x^2 + z_y^2}\,dA$ where the region of integration is the circle $x^2 + y^2 = 1$. Switching to polar gives

$$\int_0^{2\pi} \int_0^1 \left(1 - r^2\right) \sqrt{1 + 4r^2} r dr d\theta = \left(\frac{5\sqrt{5}}{24} - \frac{11}{120} \right) 2\pi$$

6. Find the flux $\iint\limits_{S} \vec{F} \cdot \vec{n} dS$ through the surface of the plane $x + y + z = 1$ in the first quadrant over the vector field $\vec{F} =< 2x, 2y, 2z + 2 >$

Soln. The normal to the surface is given by $\vec{n} = \dfrac{< 1, 1, 1 >}{\sqrt{3}}$. For this surface $dS = \sqrt{1 + 1 + 1}\,dA$ so $\iint\limits_{S} \vec{F} \cdot \vec{n} dS = \iint_S < 2x, 2y, 2z + 2 > \cdot \dfrac{< 1, 1, 1 >}{\sqrt{3}} \sqrt{1 + 1 + 1}\,dA = \iint_S (2x + 2y + 2z + 2)\,dA$. Bringing in the surface we obtain

$$\int_0^1 \int_0^{1-x} 4\,dy\,dx = 2$$