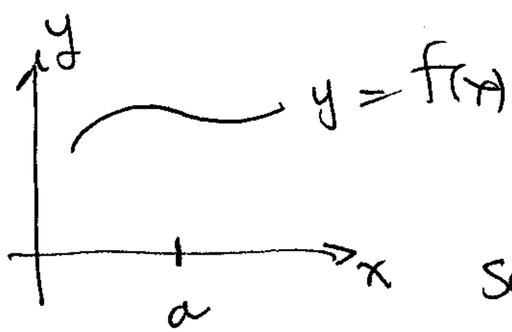
Math 1497 - Calc II

This is a continuation from Calc I. In this lecture we will go over some of the things we covered in Calc I.

Limits



so as we get close to $x = a$, what is $f(x)$ doing? is $f(x) \to L$ (#)

symbolically

$$\lim_{x \to a} f(x) = L \quad (\text{if } L \text{ exists})$$

we introduced 3 techniques to see if we can find limits

(1) graphically (picture)
(2) numerically (table)
(3) analytically

Ex: $\lim_{x\to 1} \frac{x^2-1}{x^2-3x-2} = $ "$\frac{0}{0}$"   "$\frac{0}{0}$" mean nothing

factoring $\lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-2)} = \lim_{x\to 1} \frac{x+1}{x-2} = \frac{2}{-1} = -2$

we also introduced rationalization & squeeze thm

Squeeze Thm: If $f(x) \le g(x) \le h(x)$ near $x=a$

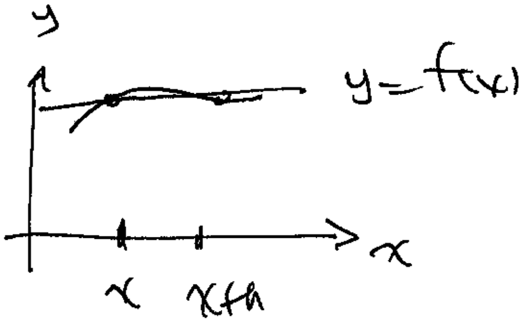
& $\lim_{x\to a} f(x) = L$ & $\lim_{x\to a} h(x) = L$

then $\lim_{x\to a} g(x) = L$

we used this to prove

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

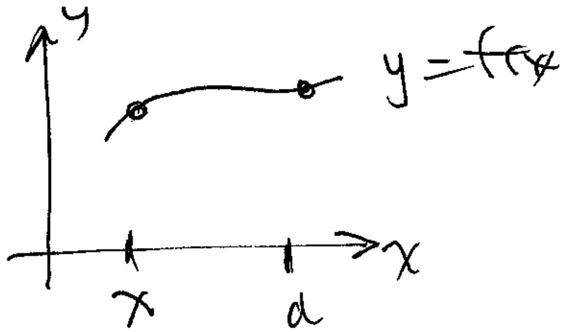
Derivatives



$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

slope of tangent

Derivative at a Point



$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Ex 2 $\quad f(x) = x^2 \qquad a = 1$

(1) $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x + h)}{\cancel{h}} = 2x$$

so $f'(x) = 2x$

(2) $$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2 \quad \text{(see Ex 1}$$

Note if $f'(x) = 2x \quad f'(1) = 2$ same

Derivative Rules $f(x)$ & $g(x)$

(1) Sum/Difference $\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{df}{dx} \pm \frac{dg}{dx}$

(2) product $\frac{d}{dx}\left(f(x)g(x)\right) = \frac{df}{dx} \cdot g + f\frac{dg}{dx}$

(3) Quotient $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$

(4) const multiple $\frac{d}{dx} cf(x) = c\frac{df}{dx}$

chain Rule

$$y = f(g(x))$$

$$y' = f'(g(x)) \cdot g'(x)$$

then we let $u = g(x)$ so $y = f(u)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Ex 3 $y = \sqrt{x^2+1}$, $u = x^2+1$, so $y = u^{1/2}$

$\frac{du}{dx} = 2x$, $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$, $\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$

Implicit Differentiation

we can take derivative even if f(x) is implicitly defined

ex 4 $x^2 - xy + y^3 = x + y - 4$

$2x - (xy' + y) + 3y^2 y' = 1 + y'$

$-xy' + 3y^2 y' - y' = 1 - 2x + y$

$(-x + 3y^2 - 1) y' = 1 - 2x + y$

$\Rightarrow \quad y' = \dfrac{1 - 2x + y}{-x + 3y^2 - 1}$ ← note: involves x & y

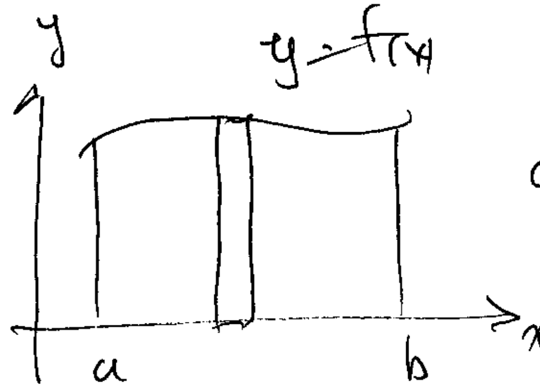
With these tools we head into applications

1) relative min/max, increasing/decreasing
2) concavity & pts of inflection
3) curve sketching

Related Rates & optimization

Integratein

we first approximated area's using rectangles and the create a Rieman Sum



(1) subdivide interval.

$a = x_0 < x_1 < \dots < x_i < x_{i+1} < \dots x_n = b$

(2) ith rectangle

$\Delta x_i = x_{i+1} - x_i$

$h_i = f(c_i)$ $\quad c_i$ in $[x_i, x_{i+1}]$

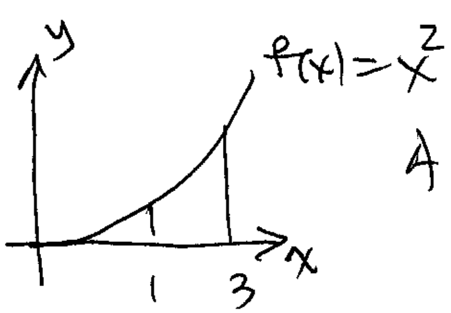
then $$\lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx$$ Definite integral

Then the fundamental Th^m of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$ $\quad$ F anti derivative

Ex



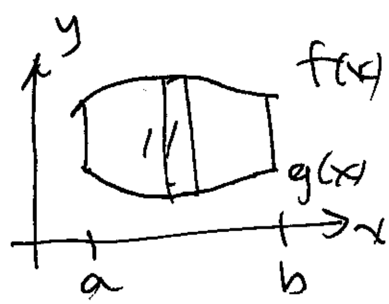
$$A = \int_1^3 x^2 dx = \frac{x^3}{3}\Big|_1^3 = \frac{3^3}{3} - \frac{1^3}{3}$$
$$= 26/3$$

We then create a table of standard integrals (see my website).

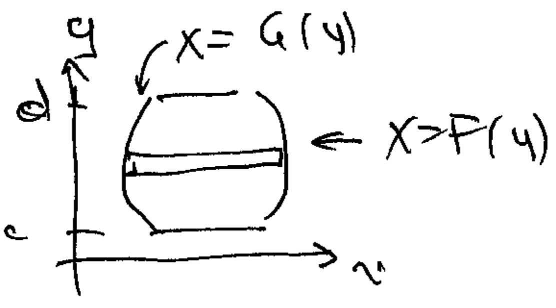
However these are very limited and we expanded integration by a technique known as "u-substitution". More on this tomorrow!

The course ended with applications of integration

Area Between Curves



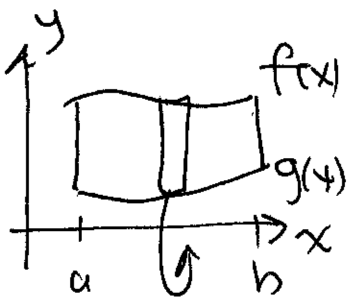
$$A = \int_a^b f(x) - g(x)\,dx$$



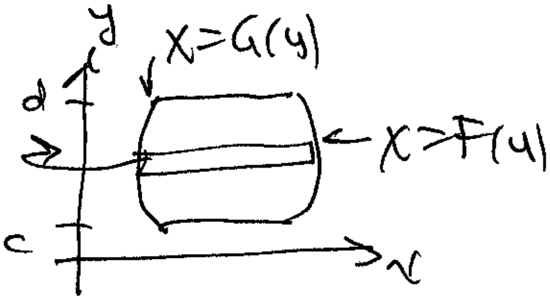
$$A = \int_c^d F(y) - G(y)\,dy$$

Volumes of Revolution

Disc Method

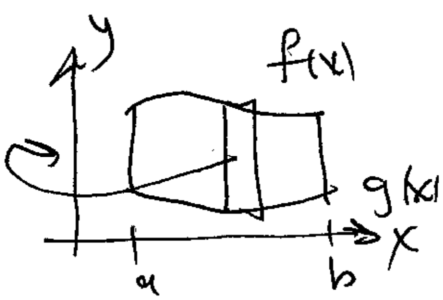


$$V = \pi \int_a^b f^2(x) - g^2(x)\,dx$$

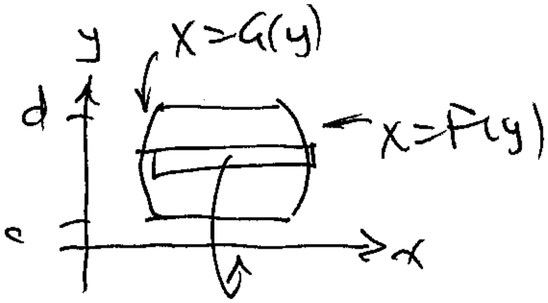


$$V = \pi \int_c^d F^2(y) - G^2(y)\,dy$$

Shell Method

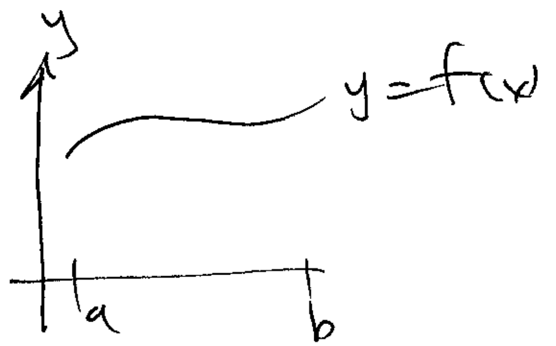


$$V = 2\pi \int_a^b x\big(f(x) - g(x)\big)\,dx$$



$$V = 2\pi \int_c^d y\big(F(y) - G(y)\big)\,dy$$

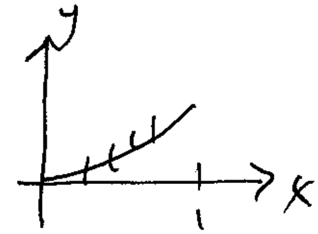
The final application is arc length s



$$s = \int_a^b \sqrt{1+y'^2}\,dx$$

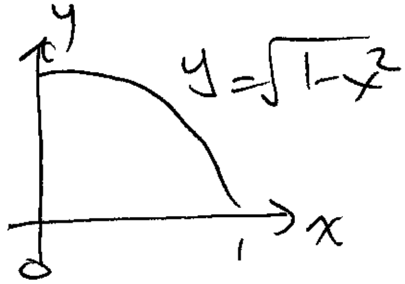
consider $y = \frac{1}{2}x^2$ on $[0,1]$ so $y' = x$

$$s = \int_0^1 \sqrt{1+x^2}\,dx$$



how do we integrate this

consider the area under $y = \sqrt{1-x^2}$ on $[0,1]$



$$A = \int_0^1 \sqrt{1-x^2}\,dx$$ how do we integrate this

so Calc 2 starts with techniques of integration.