

A Model For Reflected Light in Curved Mirrors

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1. Introduction

The study of reflections in curved mirrors has a long and extensive history. Famous Physicists and Applied Mathematicians such as Snell (1580-1626), Hooke (1635-1703), Newton (1642-1727), Huygens (1629-1695) and Maxwell (1831-1879) were early developers of the field of optics to which this particular problem belongs. The field of optics includes many physically important applications. For example, if one wishes to design a telescopic dish, then it is desirable to create the correct conical dish such that all the rays entering the dish are reflected to a common point. A second application is found in the automotive industry; in particular, the design of the headlights. Ideally, the manufacturer wishes to construct a dish such that light can be emitted from a single source, reflect from the curved surface of the headlight and emerge as a conical beam. In this paper, we will show that the ideal shape for this type of dish is that of a parabola. The paper is organized as follows: In section 2, we will formulate the mathematical model for this type of problem. Section 3 will contain our analysis where we will solve the resulting ordinary differential equation obtained in section 2. Finally, the last section will contain our conclusions.

2. The Model

Consider rays emerging from a common point (O) and reflected through a curved surface $y = f(x)$ and emerging as a straight beam (see figure 1). The point of reflection is denoted by the point $P(x,y)$ and is located on the curve $y = f(x)$ as shown below.

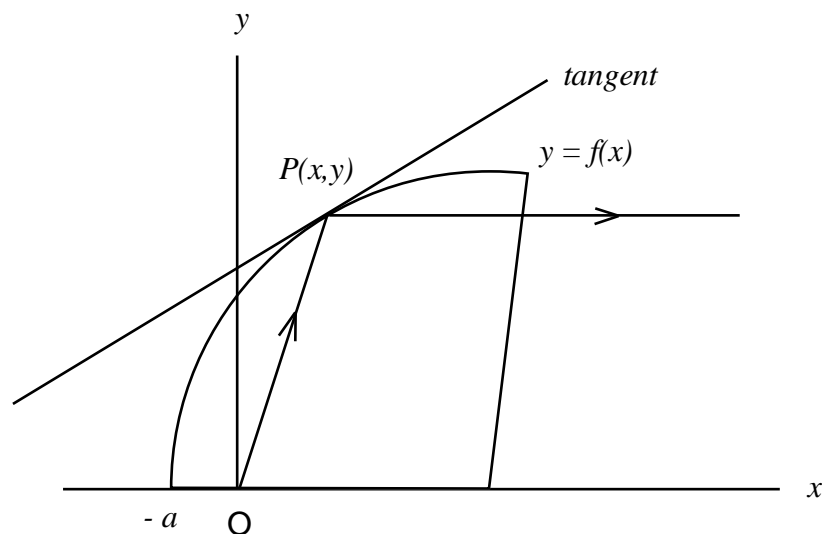


Figure 1 - reflection in a curve mirror

We will assume that the angle of incidence (θ) is equal to that of the angle of reflection relative to the equation of the tangent to the curve $y = f(x)$ at the point P (see for example

Born and Wolf [1]). We will further assume that the surface of the curve is perfectly smooth so that no dispersion of the beam itself will occur. We will denote the slope of the tangent line at P as m_1 , the slope of the incident ray as m_2 and the slope of the reflected ray as m_3 (see figure 2).

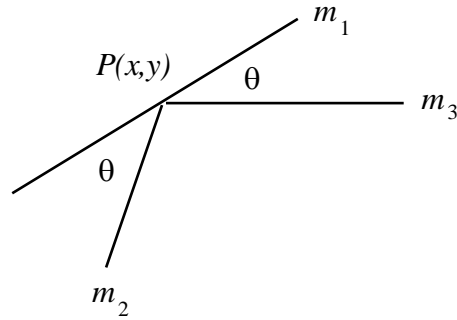


Figure 2 - reflection at the tangent line at P

From trigonometry, we have the following relationship between the angle θ between two lines and their respective slopes (m_1 and m_2) (see for example CRC Standard Mathematical Tables [2])

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}. \quad (2.1)$$

For the incident ray we have

$$m_1 = \frac{dy}{dx} \text{ and } m_2 = \frac{y - 0}{x - 0}, \quad (2.2)$$

for the slopes of the tangent line and for the line through the origin and the tangent point, respectively. Thus, we have

$$\tan \theta = \frac{\frac{y}{x} - \frac{dy}{dx}}{1 + \frac{y}{x} \frac{dy}{dx}}. \quad (2.3)$$

For the reflected ray we have $m_3 = 0$ since it is simply a horizontal line. Thus, we also have

$$\tan \theta = \frac{dy}{dx}. \quad (2.4)$$

Eliminating $\tan \theta$ from equations (2.3) and (2.4) gives

$$\frac{\frac{y}{x} - \frac{dy}{dx}}{1 + \frac{y}{x} \frac{dy}{dx}} = \frac{dy}{dx}, \quad (2.5)$$

which, upon simplifying gives

$$\frac{y}{x} \left(\frac{dy}{dx} \right)^2 + 2 \frac{dy}{dx} - \frac{y}{x} = 0. \quad (2.6)$$

The appropriate boundary condition is (see figure 1)

$$f(-a) = 0. \quad (2.7)$$

Thus, we are required to solve equation (2.6) subject to the boundary condition (2.7) (see Zill [3] for further details of this particular physical problem).

3. Mathematical Analysis

In this section we will derive the solution of the ordinary differential equation (2.6). Dividing (2.6) by y/x gives

$$\left(\frac{dy}{dx} \right)^2 + 2 \frac{x}{y} \frac{dy}{dx} - 1 = 0. \quad (3.1)$$

Completing the square in $\frac{dy}{dx}$ yields

$$\left(\frac{dy}{dx} + \frac{x}{y} \right)^2 = \frac{x^2 + y^2}{y^2}, \quad (3.2)$$

from which we can derive

$$\frac{dy}{dx} + \frac{x}{y} = \pm \frac{\sqrt{x^2 + y^2}}{y},$$

or

$$y \frac{dy}{dx} + x = \pm \sqrt{x^2 + y^2}. \quad (3.3)$$

Under the substitution

$$u = x^2 + y^2, \quad (3.4)$$

gives (3.3) as

$$\frac{du}{dx} = \pm 2\sqrt{u}, \quad (3.5)$$

which, upon integration gives

$$\sqrt{u} = \pm(x + c), \quad (3.6)$$

where c is an arbitrary constant of integration. Back substitution of (3.4) and squaring gives

$$y^2 = 2cx + c^2. \quad (3.7)$$

Employing the boundary condition (2.7) gives

$$0 = -2ca + c^2,$$

from which we deduce that $c = 2a$ since $c = 0$ would lead to the trivial solution of (2.6).

Thus, the solution of (2.6) subject to the boundary condition (2.7) is given by

$$\begin{aligned} y^2 &= 4ax + 4a^2 \\ &= 4a(x + a), \end{aligned} \quad (3.8)$$

the standard equations for a parabola with the center at $(-a, 0)$ and focus at $(0, 0)$ (see for example Larson and Hostetler [4]). A typical profile of equation (3.8) is given in figure 3 where the value of a is taken as unity.

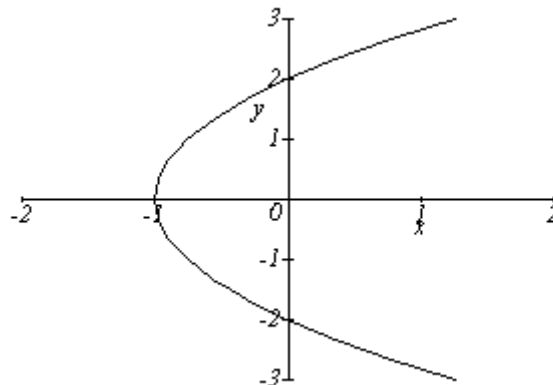


Figure 3. A typical parabolic mirror with $a = 1$

4. Conclusion

In this paper, we have considered the problem of deducing the nature of the curve such that when rays are emitted from a common point and allowed to reflect from a curve, the resulting rays will all leave in the same direction (in our case, horizontally). We obtained a mathematical model for this problem using the property that the angle of incidence and the angle of reflection of the rays are equal. From the analysis of the resulting ordinary differential equation, we deduced that the shape of the curve must be parabolic. In our model we have assumed that the mirror is perfectly smooth and that dispersion will not occur. Realistically, we would naturally have to take into account a small amount of dispersion. Furthermore, it is sometimes desirable for the beam to widen as it emerges from the mirror, i.e. to become frustum in shape. The extent of the widening naturally depends on the targeted distance of the emerging beam. Furthermore, the targeted distance would also be dependent on the strength of the light source. As this is a simplified model, we have neglected these effects.

5. References

- [1] Born, Max and Wolf, Emil, "Principles of Optics", 6th Edition, Pergamon Press Eds. (1980)
- [2] "CRC Standard Mathematical Tables" 25th Ed., CRC Press, Inc. (1978)
- [3] D. G. Zill "A first course in differential equations with applications", Prindle, Weber and Schmidt, Publishers (1979)
- [4] Larson and Hostetler "Precalculus" 4th Edition, Houghton Mifflin Company Pubs. (1997)