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Research Article

Sub implicative ideals of KU-Algebras

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Abstract

The notions of ku-sub-implicative ideals, ku-sub-commutative ideals and kp-ideals of ku-algebras are introduced. We show that a nonempty subset of a KU-algebra is a ku-sub-implicative ideal if and only if it is both a ku-sub-commutative ideal and a ku-positive implicative ideal. We discuss the relation between kp-ideal and a ku-sub-implicative ideal and a ku-sub-commutative ideal that is, in ku-algebra any kp-ideal is always ku-sub-implicative and ku-sub-commutative ideals, but the converse is not true. We give a characterization of ku-positive implicative ideals of ku-algebras. Moreover some other properties about ku-sub implicative ideals and ku-sub commutative ideals of ku-algebras are given. We give conditions for ideals to be a ku-sub-implicative ideal, ku-sub-commutative ideal and ku-positive implicative ideal. Moreover we show that any ku-sub-implicative ideal, a ku-sub-commutative ideal and kp-ideal are ideals, but the converse is not true. We verify that, in an implicative ku-algebra every ideal is a ku-sub-commutative. In the end, some algorithms for KU-algebra have been constructed.

Keywords: Ku-algebras; Ku-sub implicative ideals; Ku-sub-commutative; Ku-positive implicative; Kp-ideal.

Introduction

Iami and Iseki [1-3] in 1966 proposed the notion of BCK-algebras. Iseki [2] introduced the notion of a BCI-algebra which is a generalization of BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures including lattices and Boolean algebras. There is a great deal of literature has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. For the general development of BCK/BCI-algebras the ideal theory plays an important role. The notions of ideals in BCK-algebras and positive implicative (implicative) ideals in BCK-algebras were introduced by Iseki [1-3].The notions of commutative ideals in BCK-algebras and implicative ideals in BCK-algebras were introduced by Meng [7-11]. Prabpayak and Leerawat [15,16] introduced a new algebraic structure which is called KU -algebra . They gave the concept of homomorphisms of KU - algebras and investigated some related properties. Mostafa et al [12-14] introduced the notion of KU-ideals of KU-algebras and then they investigated several basic properties which are related to KU-ideals .The idea of sub implicative ideal was introduced by Liu and Meng [6], they established the concepts of sub-implicative ideals and sub-commutative ideals in BCI-algebras and investigated some of their properties.

The goal of this paper is to introduce the notions of ku-sub implicative, ku-positive implicative, ku-sub-commutative and kp-ideal ideals in ku-algebras and investigate some their related properties. We show that in a ku-algebra X, a nonempty subset of X is a sub-implicative ideal if and only if it is both a sub-commutative ideal and positive implicative ideal .We prove that any kp-ideal is always ku- sub-implicative and ku-sub-commutative ideals, but the converse is not true. Moreover we show that, any ku-sub implicative (ku-sub-commutative) ideal is an ideal, but the converse is not true. On the other hand the properties of ku-sub-commutative ideals of ku-algebras are given.

Preliminaries

Now we recall some known concepts related to ku-algebra from the literature which will be helpful in further study of this article.

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Definition 2.1. [15,16] Algebra(X, $*$, 0) of type (2, 0) is said to be a ku-algebra, if it satisfies the following axioms:

(ku_1) $(x*y)*[(y*z))*(x*z)]=0$,

(ku_2) $x*0=0$,

(ku_3) $0*x=x$,

(ku_4) $x*y=0$ and $y*x=0$ implies $x=y$,

(ku_5) $x*x=0$,

For all $x,y,z\in X$.

On a KU-algebra $(X,*,0)$ we can define a binary relation $\leq$ on X by putting:

$x\leq y\Leftrightarrow y*x=0$.

Thus a KU - algebra X satisfies the conditions:

$(ku_{1'})$: $(y*z)*(x*z)\leq(x*y)$

$(ku_{2'})$: $0\leq x$

$(ku_{3'})$: $x\leq y, y\leq x$ implies $x=y$,

$(ku_{4'})$: $y*x\leq x$.

Theorem 2.2. [12]: In a ku-algebra X, the following axioms are satisfied:

For all $x,y,z\in X$,

(1): $x\leq y$ imply $y*z\leq x*z$,

(2): $x*(y*z)=y*(x*z)$, for all $x,y,z\in X$,

(3): $((y*x)*x)\leq y$.

(4) $((y*x)*x)*x))=(y*x)$

Proof. Since

$(y*z)*(x*z)\leq(x*y)$,

implies $x*((y*z)*z)\leq(x*y)$, put x=0, we have $0*((y*z)*z)\leq(0*y)\Rightarrow(y*x)*x\leq y$,

then y*x≤ ((y*x)*x)*x…………..1

But,

$(y*x)*[((y*x)*x)*x)]=[(y*x)*x)]*[(y*x)*x)]=0$

$i.e$ $((y*x)*x)*x)\leq(y*x)$……………………2

From 1, 2 ,we have $((y*x)*x)*x))=(y*x)$

We will refer to X is a ku-algebra unless otherwise indicated.

Definition 2.3. [15,16] Let I be a non-empty subset of a KU-algebra X. Then I is said to be an ideal of X, if

(I_1) $0\in I$

(I_2) $\forall y,z\in X$, if $(y*z)\in I$ and $y\in I$, imply $z\in I$.

Definition 2.4. [12] Let I be a non-empty subset of a KU-algebra X. Then I is said to be an KU- ideal of X, if

(I_1) $0\in I$

(I_2) $\forall x,y,z\in X$, if $x*(y*z)\in I$ and $y\in I$, imply $x*z\in I$.

Definition 2.5. [13,14] A ku-algebra $(X,*,0)$ is said to be ***ku*** -positive implicative, if it satisfies: $(z*x)*(z*y)=z*(x*y)$, for all x,y,z in X.

Theorem 2.6.[13,14] Let $(X,*,0)$ be a ku-algebra. X is ku-positive implicative if and only if $y*x=y*(y*x)$.

Definition 2.7. [13,14] a ku-algebra $(X,*,0)$ is said to be ***ku*** - commutative if it satisfies: $\forall x,y\in X$, $(y*x)*x=(x*y)*y$.

Theorem2.8. [13,14] For a ku-algebra $(X,*,0)$, the following are equivalent:

(a) X is ***ku*** - commutative,

(b) $(y*x)*x\leq(x*y)*y$,

(c) $((x*y)*y)*((y*x)*x)=0$.

Definition 2.9. [13,14]. A KU-algebra $(X,*,0)$ is called ***ku***- implicative if $x=(x*y)*x$, for all x,y in X.

Definition 2.10. [13,14]. ku-algebra is said to be implicative if it satisfies $(x*y)*y=(x*y)*((y*x)*x))$

Definition2.11. [13,14]. ku-algebra is said to be commutative if it satisfies $x\leq y$ *implies*$(x*y)*y=x$

Lemma 2.12. [13,14]. Let X be a KU-algebra. X is ku-implicative if X is ku-positive implicative and ku-commutative.

Ku-(Subimplicative, positive implicative, subcommutative) ideals

In this section, we discuss the notion of ku (sub implicative\positive implicative\sub commutative) ideals, and then we give some characterizations of these concepts.

Definition 3.1. A non-*empty subset A of a KU-algebra X is called a ku-sub* implicative ideal of X, if $\forall x,y,z\in X$,

(1) $0\in A$

(2) $z*((x*y)*((y*x)*x))\in A$ and $z\in A$, imply $(x*y)*y\in A$.

Example 3.2. Let $X=\{0,1,2,3,4\}$ in which the operation $*$ is given by the table 1.

Table 1

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Using the

algorithms in Appendix A, then $(X,*,0)$ is a KU-Algebra.

It is easy to verify that $A=\{0,1,2,3\}$ is a ku-sub implicative ideal of X.

Theorem 3.3. Let A be an ideal of X. Then A is ***ku-*** sub implicative if and only if

(A) $((x*y)*((y*x)*x))\in A$ implies $(x*y)*y\in A$

Proof. Suppose that A is a ku-sub implicative ideal of X. For any $x,y\in X$,

If $((x*y)*((y*x)*x))\in A$, then $0*((x*y)*((y*x)*x))\in A$ and $\mathbf{0}\in A$ by (definition 3.1 – (2)). Hence (A) holds.

Conversely, suppose that an ideal A satisfies (A), For $x,y,z\in X$, if $z*((x*y)*((y*x)*x))\in A$ and $z\in A,$ (by the definition of ideals) we obtain $((x*y)*((y*x)*x))\in A$, It follows from (A) that $(x*y)*y\in A$.This mean that A is a ku-sub implicative ideal. This completes the proof.

Theorem 3.4. Any sub implicative ideal is an ideal, but the converse is not true.

Proof. Suppose that A is ku-sub implicative ideal of X and let $x=y$ in (definition 3.1 – (2)), we get

$z*((x*x)*((x*x)*x))=z*[(0*(0*x))]=z*x\in A$,

$z\in A$ imply $x\in A$. This means that A is an ideal.

The last part is shown by the following example.

Example 3.5. Let $X=\{0,1,2,3,4\}$ in which the operation $*$ is given by the table 2.

Table 2

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	1	4
2	0	1	0	3	4
3	0	0	2	0	4
4	0	1	0	3	0

Using the algorithms in Appendix A, then $(X,*,0)$ is a KU-Algebra. It is easy to verify that $A=\{0\}$ is an ideal, but not ku-sub implicative ideal of X. Since,

$0*((4*2)*((2*4)*4))\in A,0\in A$, but $(4*2)*2=2\notin A$.

Definition 3.6. Let $(X,*,0)$ be a KU-algebra, a nonempty subset A of X is said to be a ku-positive implicative ideal if it satisfies, for all x,y,z in X,

(1) $0\in A$,

(2) $z*(x*y)\in A$ and $z*x\in A$ imply $z*y\in A$.

Lemma 3.7. Any ku-positive implicative ideal is an ideal, but the converse is not true.

Proof: clear

Example. 3.8. Let $X=\{0,1,2,3,4\}$ in which the operation $*$ is given by the table in (example. **3.5.)** Then $(X,*,0)$ is a KU-Algebra. $\{0,1,3\},\{0,1,2,3\}$ are ku-positive implicative ideals of X. $\{0\},\{0,2\}$ *and* $\{0,2,4\}$ are ideals of X, but not ***ku***-positive implicative ideals.

Theorem 3.9. Let $(X,*,0)$ be a KU-algebra, if A is a ku - positive implicative ideal of X, the following are equivalent :

(a) A is a ku-positive implicative ideal of X,

(b) A is an ideal and for any x,y in X, $y*(y*x)\in A$ implies $y*x\in A$.

(c) A is an ideal and for any x,y,z in X, $z*(y*x)\in A$ implies $(z*y)*(z*x)\in A$.

(d) $0\in A$ and $z*(y*(y*x))\in A,z\in A$ implies $y*x\in A$.

Proof. ($a\Rightarrow b$).If A is a ku-positive implicative ideal of X, by Lemma 3.7. is an ideal. Suppose $y*(y*x)\in A$, since $y*y=0\in A$,by ***Definition 3.6*** (2)**,** we have $(y*x)\in A$, (b) hold.

($b\Rightarrow c$).Assume (b) and $z*(y*x)\in A$. Since

$z*((z*((z*y)*x)))=z*(((z*y)*(z*x))\leq z*(y*x)\in A,$

it follows that

$z*((z*((z*y)*x))\in A$,by (b) we have

$(z*y)*(z*x)=z*((z*y)*x)\in A$

And so (c) hold.

($c\Rightarrow d$) It clear that $0\in A$.

If $z*((y*(y*x))\in A$, $z\in A$, then

$(y*(y*(z*x))\in A$ by (c), we get

$z*(y*x)=0*(z*(y*x)=(y*y)*(z*(y*x)\in A$

Since A is an ideal and $z\in A$, then $(y*x)\in A$, and so (d) hold

($d\Rightarrow a$) First observe that if A satisfied (d), then A is an ideal of X.

In fact suppose $(y*x)\in A$ and $y\in A,$ then $(y*(0*(0*x))\in A,y\in A,$

using (d) we obtain x=$0*x\in A$, i.e A is an ideal. Next, let $z*(y*x)\in A$ and $z*y\in A$. As

$(z*y)*(z*(z*x))\leq y*(z*x)=z*(y*x)\in A,$

it follows that

$(z*y)*(z*(z*x))\in A$. Combining $(y*x)\in A$ and using (d), we have $(z*x)\in A$. This have proved A is a ku-positive implicative ideal of X.

Theorem 3.10. Any sub-implicative ideal is a positive implicative ideal, but the converse does not hold.

Proof. Assume that A is a ku-sub implicative ideal of X. It follows from (Theorem 3,4) that A is an ideal. In order to prove that A is a positive implicative *ideal from (Theorem 3,9(b))* it suffices to show that if $y*(y*x)\in A$ *then* $y*x\in A$

by (Theorem 3.3)for any $u,v\in X$, we have $((u*v)*((v*u)*u))\in A$ implies $(u*v)*v\in A$ Substituting $x=u, y*x=v$,then

$((u*v)*((v*u)*u)=\{(x*(y*x)*\{\}=(y*x)*x)*\{(y*x)\}$

$=((y*((y*x)*x)*x)=y*(y*x)\in A$

Hence if $y*(y*x)\in A$, then ,$(u*v)*v\in A$, i.e

$(((x*(y*x))*(y*x)=((y*(x*x))*(y*x)=0*(y*x)=(y*x)\in A$

Therefore A is a ku-positive implicative ideal of X .

In example 3.5. $\{0,1,3\}$ is ku-positive implicative ideal, but not ku-sub implicative ideal. This finishes the proof.

Definition 3.11. A non-empty subset A of a KU-algebra X is called a ku-sub commutative ideal of X , if

(1) $\mathbf{0}\in A$

(2) $z*\{((y*x)*x)*y)*y\}\in A$ and $z\in A$, imply $(y*x)*x\in A$.

Example *3.12*. Let $X=\{0,1,2,3\}$ in which the operation $*$ is given by the table 3.

Table 3

*	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	2	0

Using the algorithms in Appendix A, then $(X,*,0)$ is a ku-Algebra. It is easy to verify that {0} and {0,3} are all ku-sub-commutative ideals of X.

Proposition 3.13. An ideal A of X is ku-sub-commutative if and only if

(B) $((y*x)*x)*y)*y\in A$ we have $(y*x)*x\in A$.

Proof. Suppose that A is a ku-sub commutative ideal of X. For any $x,y\in X$,

If $((y*x)*x)*y)*y\in A$, then

$0*((y*x)*x)*y)*y\in A$ and $\mathbf{0}\in A$ by Definition 3.11 – (2). Hence (B) holds.

Conversely, suppose that an ideal A satisfies (B), For $x,y,z\in X$, if $z*((y*x)*x)*y)*y)\in A$ and $z\in A$, (by the definition of ideals) we obtain

$((y*x)*x)*y)*y)\in A$, It follows from (B) that $(y*x)*x\in A$.This mean that A is a ku-sub commutative ideal. This completes the proof.

Proposition 3.14. ku-sub-commutative ideal is an ideal, but the converse does not hold.

Proof. Suppose that A is ku-sub commutative ideal of X and let $x=y$ in (definition 3.11 – (2)), $z*\{((x*x)*x)*x)*x\}=z*x\in A$, $z\in A$, imply $x\in A$. This means that A is ideal.

The last part is shown by the example 3.5. $\{0,1,3\}$ is ideal, but not ku-sub commutative ideal. This finishes the proof.

Definition 3.15. A nonempty subset A of a KU-algebra X is called a kp-ideal of X if it satisfies

(1) $\mathbf{0}\in A$,

(2) $(z*y)*(z*x)\in A$, $y\in A\Rightarrow x\in A$

Proposition 3.16. kp-ideal ideal is an ideal, but the converse does not hold

Proof: clear

Theorem 3.17. Any kp-ideal is a ku- sub-implicative ideal, but the converse is not true.

Proof. Suppose that A is kp-ideal, then A is ideal . Now we show

$(x*y)*((y*x)*x))\in A$ implies $(x*y)*y\in A$

$[(x*y)*((y*x)*x))]*[(x*y)*y]\leq y*((y*x)*x)=(y*x)*(y*x)=0\in A$

We have $(x*y)*((y*x)*x))\in A$ it follows that $(x*y)*y\in A$. By (Theorem 3.3) means that A is ku-sub implicative. The last part is shown by the following example

Example 3.18. Let $X=\{\mathbf{0,1,2,3,4}\}$ in which the operation $*$ is given by the table 4.

Table 4

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	4
2	0	0	0	3	4
3	0	0	1	0	4
4	0	0	0	0	0

Using the algorithms in Appendix A, then $(X,*,0)$ is a ku-Algebra. It is easy to verify that $A=\{\mathbf{0,1,2,3}\}$ is a ku-sub implicative ideal of X , but not kp-ideal, since $(4*0)*(4*4)=0\in A$, $0\in A\Rightarrow 4\notin A$

Theorem 3.19. Any kp-ideal is a ku-sub-commutative ideal, but the converse is not true.

Proof. Suppose that A is kp-ideal, then A is ideal. Now we show

$((y*x)*x)*y)*y \in A$ implies $(y*x)*x \in A$.

$\{((y*x)*x)*y)*y\}*[(y*x)*x]=$

$(y*x)*[\{(y*x)*x)*y)*y\}*x] \leq \{(y*x)*x)*y)*y\}*y=0 \in A$

We have $((y*x)*x)*y)*y \in A$ it follows that, by (Proposition 3.13) it means that A is ku-sub-commutative.

In example 3.12. It is easy to verify that {0,3} is ku-sub-commutative ideals

but not kp-ideal , since $(2*3)*(2*2)=0 \in A$, $3 \in A \Rightarrow 2 \notin A$

Corollary 3.20. In an implicative KU-algebra every ideal is an ku- sub-commutative

Proof. The proof is straightforward.

Conclusions

In the present work the the concepts of ku-sub implicative, ku-positive implicative, ku-sub-commutative and kp-ideals in ku-algebras is introduced and some their related properties is investigated. We show that in a ku-algebra X, a nonempty subset of X is a sub-implicative ideal if and only if it is both a sub-commutative ideal and positive implicative ideal .We prove that any kp-ideal is always an ku- sub-implicative ideal and an ku-sub-commutative ideal, but the converse is not true. Moreover we prove that any ku-sub implicative (ku-sub-commutative) ideal is an ideal, but the converse is not true.

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Conflicts of Interest

State any potential conflicts of interest here or the authors declare no conflict of interest.

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