Research Article

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Some Equal-area, Conformal and Conventional Map Projections: A Tutorial Review

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Abstract: Map projections have been widely used in many areas such as geography, oceanography, meteorology, geology, geodesy, photogrammetry and global positioning systems. Understanding different types of map projections is very crucial in these areas. This paper presents a tutorial review of various types of current map projections such as equal-area, conformal and conventional. We present these map projections from a model of the Earth to a flat sheet of paper or map and derive the plotting equations for them in detail. The first fundamental form and the Gaussian fundamental quantities are defined and applied to obtain the plotting equations and distortions in length, shape and size for some of these map projections.

Keywords: Conformal, Distortion, Equal-area, Gaussian Fundamental Form, Geographical Information System, Map Projection, Tissot's indicatrix

1 Introduction

A *map projection* is a systematic transformation of the latitudes and longitudes of positions on the surface of the Earth to a flat sheet of paper, a map. More precisely, a map projection requires a transformation from a set of two independent coordinates on a model of the Earth (the latitude ϕ and longitude λ) to a set of two independent coordinates on the map (e. g., the Cartesian coordinates x and y), i. e., a transformation matrix T such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} \phi \\ \lambda \end{bmatrix}. \tag{1}$$

However, since we deal with partial derivative and fundamental quantities (to be defined in Section 2), it is impossible to find such a transformation explicitly.

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There are a number of techniques for map projection, yet in all of them distortion occurs in length, angle, shape, area or in a combination of these. Carl Friedrich Gauss showed that a sphere's surface cannot be represented on a map without distortion (cf., [8]).

A *terrestrial globe* is a three dimensional scale model of the Earth that does not distort the real shape and the real size of large futures of the Earth. The term *globe* is used for those objects that are approximately spherical. The equation for spherical model of the Earth with radius R is

$$\frac{x^2}{R^2} + \frac{y^2}{R^2} + \frac{z^2}{R^2} = 1. \tag{2}$$

An oblate *ellipsoid* or *spheroid* is a quadratic surface obtained by rotating an ellipse about its minor axis (the axis that passes through the north pole and the south pole). The shape of the Earth is appeared to be an oblate ellipsoid (mean Earth ellipsoid), and the geodetic latitudes and longitudes of positions on the surface of the Earth coming from satellite observations are on this ellipsoid. The equation for spheroidal model of the Earth is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1, \tag{3}$$

where a is the semi-major axis, and b is the semi-minor axis of the spheroid of revolution.

The spherical representation of the Earth (terrestrial globe) must be modified to maintain accurate representation of either shape or size of the spheroidal representation of the Earth. We discuss about these two representations in Section 3.

There are three major types of map projections:

1. **Equal-area projections.** These projections preserve the area (the size) between the map and the model of the Earth. In other words, every section of the map keeps a constant ratio to the area of the Earth that represents. Some of these projections are Albers with one or two standard parallels [6] (the conical equal-area), the Bonne [8], the azimuthal and Lambert cylindrical equal-area [6, 8] which are best applied to a local area of the Earth, and some of them are world maps such

as the sinusoidal [8], the Mollweide [8, 12], the parabolic [13], the Hammer-Aitoff [8, 12], the Boggs eumorphic [13], and Eckert IV [11]. As an application of equal-area projections, we can mention for instance Lambert azimuthal projection [6, 8] which is often used by researchers in structural geology to plot crystallographic axes and faces, lineation and foliation in rocks, slickensides in faults, and other linear and planar features.
2. **Conformal projections.** These projections, while not so useful for portraying large areas, are very important in surveying and mapping, as angles are truly preserved. These projections include the Mercator, the Lambert conformal with one standard parallel, and the stereographic (e. g., [6, 8]. These projections are only applicable to limited areas on the model of the Earth for any one map. Conformal projections have applications in topography, certain kinds of navigation and geographical information system (GIS). Lambert conformal conic and Mercator conformal cylindrical are commonly used for GIS. Google and Bing Maps use a variant of the Mercator conformal projection called Web Mercator or Google Web Mercator. This for instance shows Greenland as large as Australia, however, in reality Australia is more than three and half times larger than Greenland (e. g., [4]).
3. **Conventional projections.** These projections are neither equal-area nor conformal, and they are designed based on some particular applications, for instance in GIS and on web pages. Some examples are the simple conic [6], the gnomonic [8], the azimuthal equidistant [9], the Miller [12], the polyconic [13], the Robinson [8], and the Plate Carree projections [8, 9]. The Plate Carree was commonly used in the past since it has a very simple calculation before computers allowed for complex calculations.

In this paper, we only show the derivation of plotting equations on a map for the Mercator cylindrical conformal and Lambert cylindrical equal-area for a spherical model of the Earth (Section 4), the Albers with one standard parallel and the azimuthal for a spherical model of the Earth and the Lambert conformal with one standard parallel for a spheroidal model of the Earth (Section 5), the sinusoidal (Section 6), the simple conic and the plate carree projections (Section 7). The methods of obtaining other projections are similar to these projections, and we refer the reader to [3, 6, 8, 9].

In Section 8, the equations for distortions of length, area and angle are derived. As an example, the distortions in length for the Albers projection and in length and area for the Mercator projection are calculated [8, 9, 15].

2 Mathematical Fundamentals of Map Projections

In this section, we derive the first fundamental form for a general surface that completely describes the metric properties of the surface, and it is a key in map projection. Furthermore, we discuss about Tissot's indicatrices (Tissot's ellipses) that usually depict on the maps to show the various types of distortions. This section is mainly based on [5, 6, 8, 9, 12, 15].

2.1 Differential Geometry of a General Surface and First Fundamental Form

The parametric representation of a surface requires two parameters. Let α and β be these two parameters. For instance, for a particular surface such as the model of the Earth, these two parameters are the latitude and longitude. The vector at any point P on a surface is given by $\vec{r} = \vec{r}(\alpha, \beta)$. If either of parameters α or β is held constant and the other one is varied, a space curve results (cf., Figure 1).

The tangent vectors to α-curve and β-curve at point P are respectively as follows:

$$\vec{a} = \frac{\partial \vec{r}}{\partial \alpha}, \qquad \vec{b} = \frac{\partial \vec{r}}{\partial \beta}. \tag{4}$$

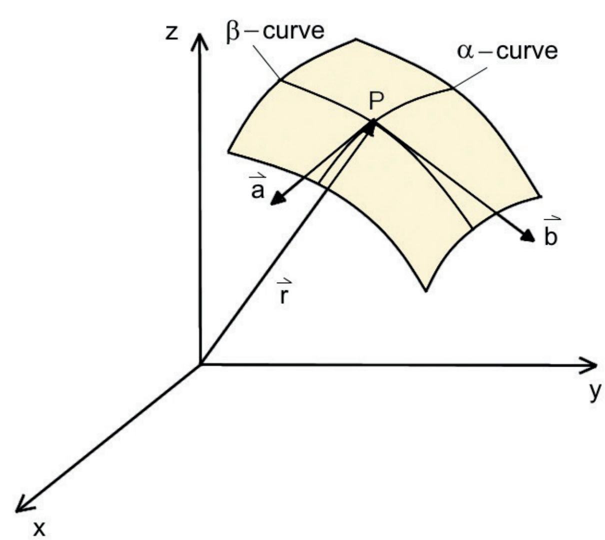

Figure 1: Geometry for parametric curves.

The total differential of $\vec{r}$ is

$$d\vec{r} = \vec{a}d\alpha + \vec{b}d\beta. \tag{5}$$

The first fundamental form (e. g., [8]) is defined as the dot product of Equation (5) with itself:

$$\begin{aligned}(ds)^2 &= d\vec{r}\cdot d\vec{r} = \left(\vec{a}d\alpha + \vec{b}d\beta\right)\cdot\left(\vec{a}d\alpha + \vec{b}d\beta\right)\\ &= E(d\alpha)^2 + 2Fd\alpha d\beta + G(d\beta)^2,\end{aligned} \tag{6}$$

where $E = \vec{a}\cdot\vec{a}$, $F = \vec{a}\cdot\vec{b}$ and $G = \vec{b}\cdot\vec{b}$ are known as the Gaussian fundamental quantities.

– From Equation 6, the distance between two arbitrary points P_1 and P_2 on the surface can be calculated:

$$\begin{aligned}s &= \int_{P_1}^{P_2}\sqrt{E(d\alpha)^2 + 2Fd\alpha d\beta + G(d\beta)^2}\\ &= \int_{P_1}^{P_2}\sqrt{E + 2F\left(\frac{d\beta}{d\alpha}\right) + G\left(\frac{d\beta}{d\alpha}\right)^2}\,d\alpha.\end{aligned} \tag{7}$$

– The angle between $\vec{a}$ and $\vec{b}$ is simply given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{F}{\sqrt{EG}}. \tag{8}$$

– Incremental area is the magnitude of the cross product of $\vec{a}d\alpha$ and $\vec{b}d\beta$, i. e.,

$$\begin{aligned}dA &= \left|\vec{a}d\alpha \times \vec{b}d\beta\right| = \left|\vec{a}d\alpha\right|\cdot\left|\vec{b}d\beta\right|\sin\theta\\ &= |\vec{a}|\cdot|\vec{b}|\sin\theta d\alpha d\beta\\ &= \sqrt{E}\sqrt{G}\sqrt{1-\cos^2\theta d\alpha d\beta}\\ &= \sqrt{EG}\sqrt{\frac{EG-F^2}{EG}}d\alpha d\beta \text{ from Equation (8)}\\ &= \sqrt{EG-F^2}d\alpha d\beta.\end{aligned} \tag{9}$$

Since we are dealing with latitudes and longitudes on a spherical or spheroidal model of the Earth, the vectors $\vec{a}$ and $\vec{b}$ are orthogonal (meridians are normal to equator parallels). Also, in maps, we are dealing with the polar and Cartesian coordinate systems in which their axes are perpendicular. Thus, from Equation (8), because $90° = 0$, one obtains $F = 0$.

Therefore, the first fundamental form Equation (6) in map projection will be deduced to the following form:

$$(ds)^2 = E(d\alpha)^2 + G(d\beta)^2. \tag{10}$$

Example 1 The first fundamental form for a planar surface

1. in the Cartesian coordinate system (a cylindrical surface) is $(ds)^2 = (dx)^2 + (dy)^2$, where $E = G = 1$,
2. in the polar coordinate system (a conical surface) is $(ds)^2 = (dr)^2 + r^2(d\theta)^2$, where $E = 1$ and $G = r^2$,
3. in the spherical model of the Earth, Equation (2), is $(ds)^2 = R^2(d\phi)^2 + R^2\cos^2\phi(d\lambda)^2$, where $E = R^2$ and $G = R^2\cos^2\phi$, and
4. in the spheroidal model of the Earth, Equation (3), is $(ds)^2 = M^2(d\phi)^2 + N^2\cos^2\phi(d\lambda)^2$, where $E = M^2$ and $G = N^2\cos^2\phi$ in which M is the radius of curvature in meridian and N is the radius of curvature in prime vertical which are both functions of ϕ:

$$M = \frac{a\left(1-e_{ab}^2\right)}{\left(1-e_{ab}^2\sin^2\phi\right)^{1.5}}, \quad N = \frac{a}{\left(1-e_{ab}^2\sin^2\phi\right)^{0.5}},$$
$$e_{ab}^2 = \frac{a^2-b^2}{a_2}. \tag{11}$$

See Figure 2, and [8, 15] for the derivations of M and N.

2.2 Transformation Equations of Map Projections

Suppose that $\alpha = \phi$ and $\beta = \lambda$ are the parameters of the model of the Earth with the fundamental quantities e, f and g given by $e = R^2$, $f = 0$, $g = R^2\cos^2\phi$ for the spherical model of the Earth, or given by $e = M^2$, $f = 0$ and $g = N^2\cos^2\phi$ for the spheroidal model of the Earth (cf., Example 1).

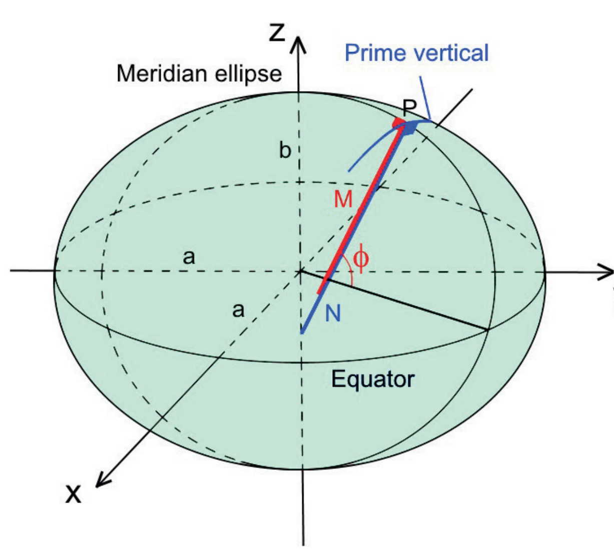

Figure 2: Geometry for the spheroidal model of the Earth, $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$.

Consider a two-dimensional projection with parametric curves defined by the parameters u and v. For instance, for the polar or conical coordinates, we have $u = r$ and $v = \theta$. Let E', F' and G' be its fundamental quantities. Also, assume that on the plotting surface a second set of parameters, x and y, with the fundamental quantities E, F and G.

The relationship between the two sets of parameters on the plane is given by

$$x = x(u, v), \qquad y = y(u, v). \tag{12}$$

As an example, $x = r\cos\theta$ and $y = r\sin\theta$ for the polar and Cartesian coordinates.

The relationship between the parametric curves ϕ, λ, u and v is

$$u = u(\phi, \lambda), \qquad v = v(\phi, \lambda). \tag{13}$$

Equation (13) must be unique and reversible, i. e., a point on the Earth must represent only one point on the map and vice versa. From Equations (12) and (13), we have

$$x = x\big(u(\phi, \lambda), v(\phi, \lambda)\big), \qquad y = y\big(u(\phi, \lambda), v(\phi, \lambda)\big). \tag{14}$$

From the definition of the Gaussian first fundamental quantities, we have

$$\begin{aligned} E &= \vec{a}\cdot\vec{a} = \left(\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi}\right)\cdot\left(\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi}\right) = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2, \\ F &= \vec{a}\cdot\vec{b} = \left(\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi}\right)\cdot\left(\frac{\partial x}{\partial \lambda}, \frac{\partial y}{\partial \lambda}\right) = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \lambda}, \\ G &= \vec{b}\cdot\vec{b} = \left(\frac{\partial x}{\partial \lambda}, \frac{\partial y}{\partial \lambda}\right)\cdot\left(\frac{\partial x}{\partial \lambda}, \frac{\partial y}{\partial \lambda}\right) = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2. \end{aligned} \tag{15}$$

Note that in here α and β in Equation (4) are replaced by ϕ and λ, respectively. Similarly, we have

$$\begin{aligned} E' &= \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right)\cdot\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right) = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2, \\ F' &= \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right)\cdot\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right) = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v}, \\ G' &= \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right)\cdot\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right) = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2. \end{aligned} \tag{16}$$

As we mentioned earlier, since we are dealing with orthogonal curves, $f = F = F' = 0$. Using this fact and Equations (14), (15) and (16), the following relation can be derived (see [8, Chapter 2]):

$$E = \left(\frac{\partial u}{\partial \phi}\right)^2 E' + \left(\frac{\partial v}{\partial \phi}\right)^2 G', \qquad G\left(\frac{\partial u}{\partial \lambda}\right)^2 E' + \left(\frac{\partial v}{\partial \lambda}\right)^2 G'. \tag{17}$$

From Equation (9), a mapping from the Earth to the plotting surface preserves area if (e. g., [8]):

$$eg = EG. \tag{18}$$

From Equations (15), (16), (17) and using $F = F' = 0$, one obtains

$$EG = J^2 \cdot E'G', \qquad J = \begin{vmatrix} \dfrac{\partial u}{\partial \phi} & \dfrac{\partial u}{\partial \lambda} \\ \dfrac{\partial v}{\partial \phi} & \dfrac{\partial v}{\partial \lambda} \end{vmatrix}, \tag{19}$$

where J is the Jacobian determinant of the transformation from the coordinate set ϕ and λ to the coordinate set u and v.

By a theorem of differential geometry (e. g., [8]), a mapping for the orthogonal curves is conformal if and only if

$$\frac{E}{e} = \frac{G}{g}. \tag{20}$$

2.3 Tissot's Indicatrix

Suppose that a terrestrial globe is covered with infinitesimal circles. In order to show distortions in a map projection, one may look at the projection of these circles in a map which are ellipses whose axes are the two principal directions along which scale is maximal and minimal at that point on the map. This mathematical contrivance is called *Tissot's indicatrix* (e. g., [6, 12]).

Let a and b respectively be the semi-major and semi-minor axes of the projected ellipse on a map with coordinates x ad y. It is shown in [12] that

$$a = 0.5\left(\sqrt{h^2 + k^2 + 2hk\sin\vartheta} + \sqrt{h^2 + k^2 - 2hk\sin\vartheta}\right), \tag{21}$$

$$b = 0.5\left(\sqrt{h^2 + k^2 + 2hk\sin\vartheta} - \sqrt{h^2 + k^2 - 2hk\sin\vartheta}\right), \tag{22}$$

where h and k represent scales along the meridian and parallel for a given point respectively, and υ is the angular deformation:

$$h = \frac{1}{R}\sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2}, \tag{23}$$

$$k = \frac{1}{R\cos\phi}\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}, \tag{24}$$

$$\sin\vartheta = \frac{1}{R^2 hk\ \cos\phi}\left(\frac{\partial y}{\partial \phi}\frac{\partial x}{\partial \lambda} - \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \lambda}\right), \tag{25}$$

where ϕ, λ, R are latitude, longitude and the radius of the globe, respectively. Moreover, the maximum angular distortion denoted by ω can be calculated as (cf., [12]):

$$\omega = 2\arcsin\left(\frac{a-b}{a+b}\right). \tag{26}$$

Usually Tissot's indicatrices are placed across a map along the intersections of meridians and parallels to the equator, and they provide a good tool to calculate the magnitude of distortions at those points (the intersections). In an equal-area projection, Tissot's indicatrices (Tissot's ellipses) change shape, whereas their areas remain the same. In conformal projection, the shape of Tissot's indicatrices preserves (i. e., $a = b$ in Equation (26), and so $\omega = 0$), but their area varies. In conventional projection, both shape and area of Tissot's indicatrices vary.

3 Projection from an Ellipsoid to a Sphere

In this section, we describe how much the latitudes and longitudes of a spheroidal model of the Earth will be affected once they are transformed to a spherical model, i. e., how much distortion in shape and size happens when one projects a spheroidal model of the Earth to a spherical model [3, 8, 9, 14]. We distinguish two cases, equal-area transformation and conformal transformation.

Case 1. A spherical model of the Earth that has the same surface area as that of the reference ellipsoid is called the *authalic sphere* [8]. This sphere may be used as an intermediate step in the transformation from the ellipsoid to the mapping surface.

Let R_A, ϕ_A and λ_A be the authalic radius, latitude and longitude, respectively. Also, let ϕ and λ be the geodetic latitude and longitude, respectively. From Example 1, we have $e = M^2$, $g = N^2\cos^2\phi$, $E' = R^2{}_A$ and $G' = R^2{}_A\cos^2\phi_A$. By Equations (18) and (19),

$$M^2N^2\cos^2\phi = R_A^4\cos^2\phi_A\begin{vmatrix} \frac{\partial\phi_A}{\partial\phi} & \frac{\partial\phi_A}{\partial\lambda} \\ \frac{\partial\lambda_A}{\partial\phi} & \frac{\partial\lambda_A}{\partial\lambda}\end{vmatrix}^2. \tag{27}$$

In the transformation from the ellipsoid to the authalic sphere, longitude is invariant, i. e., $\lambda = \lambda_A$. Moreover, ϕ_A is independent of λ_A and so λ. Thus Equation (27) reduces to

$$M^2N^2\cos^2\phi = R_A^4\cos^2\phi_A\begin{vmatrix} \frac{\partial\phi_A}{\partial\phi} & 0 \\ 0 & 1\end{vmatrix}^2. \tag{28}$$

Substitute the values of M and N given by Equation (11) into Equation (28) to obtain

$$\frac{a^2\left(1-e_{ab}^2\right)}{\left(1-e_{ab}^2\sin^2\phi\right)^2}\cos\phi d\phi = R_A^2\cos\phi_A d\phi_A. \tag{29}$$

Integrating the left hand side of Equation (29) from 0 to ϕ (using binary expansion), and the right hand side from 0 to ϕ_A, one obtains

$$R_A^2\sin\phi_A = a^2\left(1-e_{ab}^2\right)\left(\sin\phi + \frac{2}{3}e_{ab}^2\sin^3\phi + \frac{3}{5}e_{ab}^4\sin^5\phi + \frac{4}{7}e_{ab}^6\sin^7\phi + \ldots\right). \tag{30}$$

Assuming $\phi_A = 90°$ when $\phi = 90°$, Equation (30) gives:

$$R_A^2 = a^2\left(1-e_{ab}^2\right)\left(1 + \frac{2}{3}e_{ab}^2 + \frac{3}{5}e_{ab}^4 + \frac{4}{7}e_{ab}^6 + \ldots\right). \tag{31}$$

Substituting Equation (31) into Equation (30), one obtains

$$\sin\phi_A = \sin\phi\left(\frac{1 + \frac{2}{3}e_{ab}^2\sin^2\phi + \frac{3}{5}e_{ab}^4\sin^4\phi + \frac{4}{7}e_{ab}^6\sin^6\phi + \cdots}{1 + \frac{2}{3}e_{ab}^2 + \frac{3}{5}e_{ab}^4 + \frac{4}{7}e_{ab}^6 + \cdots}\right). \tag{32}$$

Since the eccentricity e_{ab} is a small number, the above series are convergent. The relation between authalic and geodetic latitudes is equal at latitudes 0° and 90°, and the difference between them at other latitudes is about

0°.1 for the WGS-84 spheroid (see [8] for the definitions of the WGS-84 and WGS-72 spheroids).

Example 2

1. For the WGS-72 spheroid with $a \approx 6{,}378{,}135$ m and $e_{ab} \approx 0.081818$, the radius of the authalic sphere is

$$R_A \approx a\sqrt{\left(1-e_{ab}^2\right)\left(1+\frac{2}{3}e_{ab}^2+\frac{3}{5}e_{ab}^4\right)} \approx 6{,}371{,}004 \text{ m}. \quad (33)$$

2. For the I. U. G. G spheroid (cf., [8]) with $f=(a-b)/a \approx 1/298.275$, we have $e_{ab}=2f-f^2 \approx 0.0066944$, and from Equation (32), for geodetic latitude $\phi=45°$, we have $\sin\phi_A \approx 0.70552$ which gives $\phi_A \approx 44°.8713$.

Case 2. A conformal sphere is an sphere defined for conformal transformation from an ellipsoid, and similar to the authalic sphere may be used as an intermediate step in the transformation from the reference ellipsoid to a mapping surface.

Let R_c, ϕ_c and λ_c be the conformal radius, latitude and longitude for the conformal sphere, respectively. Let e and g be the same fundamental quantities as Case 1, and $E'=R_c^2$ and $G'=R_c^2\cos^2\phi_c$. Also, let $\phi_c=\phi_c(\phi)$ and $\lambda_c=\lambda$. Thus, from Equation (17),

$$E=\left(\frac{\partial\phi_c}{\partial\phi}\right)^2 E'+\left(\frac{\partial\lambda_c}{\partial\phi}\right)^2 G'=\left(\frac{\partial\phi_c}{\partial\phi}\right)^2 E',$$
$$G=\left(\frac{\partial\phi_c}{\partial\lambda}\right)^2 E'+\left(\frac{\partial\lambda_c}{\partial\lambda}\right)^2 G'=G'. \quad (34)$$

Combining Equations (20) and (34), one obtains

$$\frac{\left(\frac{\partial\phi_c}{\partial\phi}\right)^2 R_c^2}{M^2}=\frac{R_c^2\cos^2\phi_c}{N^2\cos^2\phi}, \quad (35)$$

that after integrating and simplifying with the condition $\phi_c=0$ for $\phi=0$, it gives

$$\tan\left(\frac{\phi_c}{2}+\frac{\pi}{4}\right)=\tan\left(\frac{\phi}{2}+\frac{\pi}{4}\right)\left(\frac{1-e_{ab}\sin\phi}{1+e_{ab}\sin\phi}\right)^{\frac{e_{ab}}{2}}. \quad (36)$$

One can calculate ϕ_c from Equation (36) which is a function of geodetic latitude ϕ. Also, it can be shown that $R_c=\sqrt{MN}$ for a given latitude ϕ which in this case $\phi=\pi/2$. We refer the reader to [8, Chapter 5] for the derivation.

4 Mercator and Lambert Cylindrical Projections

In this section, by an elementary method, we show the cylindrical method that Mercator used to map from a spherical model of the Earth to a flat sheet of paper and derive its plotting equations. Also, we give the plotting equations for the Lambert cylindrical equal-area projection. In the following sections, however, we apply the mathematical equations of map projections described in Section 2 to obtain the plotting equations. This section is based mostly on [7].

Let S be the globe, and C be a circular cylinder tangent to S along the equator, see Figure 3. Projecting S along the rays passing through the center of S onto C, and unrolling the cylinder onto a vertical strip in a plane is called *central cylindrical projection*. Clearly, each meridian on the sphere is mapped to a vertical line to the equator, and each parallel of the equator is mapped onto a circle on the cylinder and so a line parallel to the equator on the map. All methods discussed in this section and other sections are about central projection, i. e., rays pass through the center of the Earth to a cone or cylinder. Methods for those projections that are not central are similar to central projections (see [8, 9]).

Let w be the width of the map. The scale of the map along the equator is $s=w/(2\pi R)$ that is the ratio of size of objects drawn in the map to actual size of the object it represents. The scale of the map usually is shown by three methods: arithmetical (e. g. 1:6,000,000), verbal (e. g. 100 miles to the inch) or geometrical.

At latitude ϕ, the parallel to the equator is a circle with circumference $2\pi R\cos\phi$, so the scale of the map at this latitude is

$$s_h=\frac{w}{2\pi R\cos\phi}=s\sec\phi, \quad (37)$$

where the subscript h stands for horizontal.

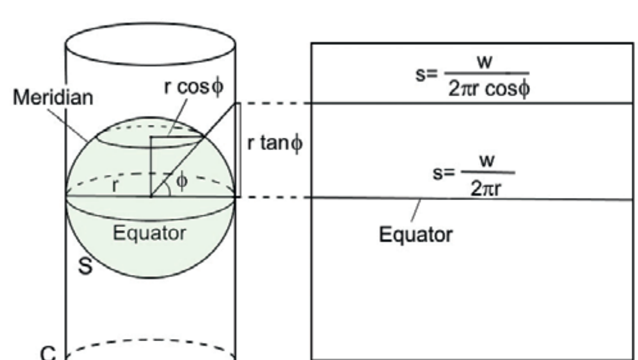

Figure 3: Geometry for the cylindrical projection.

Assume that ϕ and λ are in radians, and the origin in the Cartesian coordinate system corresponds to the intersection of the Greenwich meridian ($\lambda = 0$) and the equator ($\phi = 0$). Then every cylindrical projection is given explicitly by the following equations

$$x = \frac{w\lambda}{2\pi}, \qquad y = f(\phi). \tag{38}$$

For instance, it can be seen from Figure 3 that a central cylindrical projection is given by

$$x = \frac{w\lambda}{2\pi}, \qquad y = r\tan\phi, \tag{39}$$

where for a map of width w, a globe of radius $r = w/(2\pi)$ is chosen.

In a globe, the arc length between latitudes of ϕ and ϕ_1 (in radians) along a meridian is

$$2\pi R \cdot \frac{\phi_1 - \phi}{2\pi} = R(\phi_1 - \phi), \tag{40}$$

and the image on the map has the length $f(\phi_1) - f(\phi)$. So the overall scale factor of this arc along the meridian when ϕ_1 gets closer and closer to ϕ is

$$s_v = \frac{1}{R} f'(\phi) = \frac{1}{R}\lim_{\phi_1 \to \phi}\frac{f(\phi_1) - f(\phi)}{\phi_1 - \phi}, \tag{41}$$

where the subscript υ stands for vertical.

The goal of Mercator was to equate the horizontal scale with vertical scale at latitude ϕ, i. e., $s_h = s_\upsilon$. Thus, from Equations (37) and (41),

$$f'(\phi) = \frac{w}{2\pi}\sec\phi. \tag{42}$$

Mercator was not be able to solve Equation (42) precisely because logarithms were not invented! But now, we know that the following is the solution to Equation (42) (use $f(0) = 0$ to make the constant coming out from the integration equal to zero),

$$y = f(\phi) = \frac{w}{2\pi}\ln\left|\sec\phi + \tan\phi\right|. \tag{43}$$

Thus, the equations for the Mercator conformal projection (central cylindrical conformal mapping) are

$$x = \frac{w\lambda}{2\pi}, \qquad y = \frac{w}{2\pi}\ln\left|\sec\phi + \tan\phi\right|. \tag{44}$$

Figure 4 shows the Mercator projection with Tissot's indicatrices that do not change their shapes (all of them are circles indicating a conformal projection) while their areas increase toward the poles. More precisely, it can be easily verified from Equations (23), (24), (25) and (44) that $h = k$, $\sin\upsilon = 1$, and so $a = b = w/(2\pi R\cos\phi) = s/\cos\phi$ (cf., Equations (21) and (22)), and so ω in Equation (26) will be zero.

Now if the goal is preserving size rather than shape, then we would make the horizontal and vertical scaling reciprocal, so the stretching in one direction will match shrinking in the other. Thus, from Equations (37) and (41), we obtain $f'(\phi)\sec\phi = c$ or

$$f'(\phi) = c\cos\phi, \tag{45}$$

where c is a constant. From Equations (42) and (45), we can choose c in such away that for a given latitude, the map also preserves the shape in that area. For instance if $\phi = 0$, then we choose $c = w/(2\pi)$, and so the map near equator is conformal too. Hence, the equations for the cylindrical equal-area projection (one of Lambert's maps) are

$$x = \frac{w\lambda}{2\pi}, \qquad y = \frac{w}{2\pi}\sin\phi. \tag{46}$$

Figure 5 shows the Lambert projection with Tissot's indicatrices that have the same areas (indicating an equal-area projection) while their shapes change toward the poles.

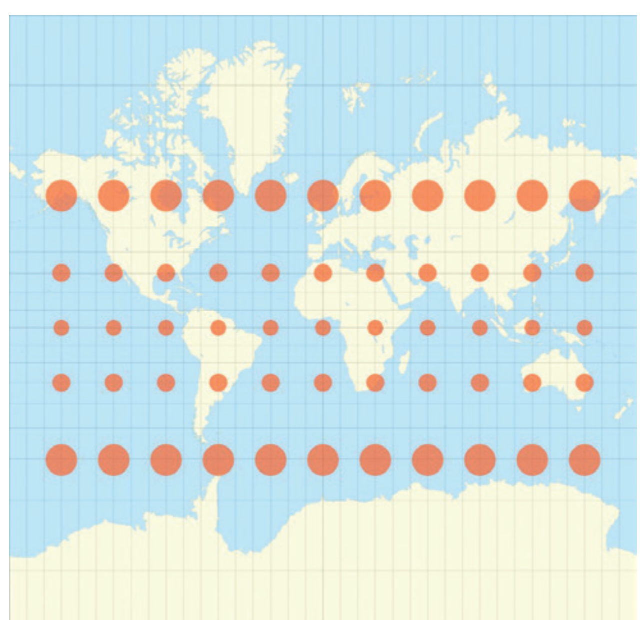

Figure 4: The Mercator conformal map.

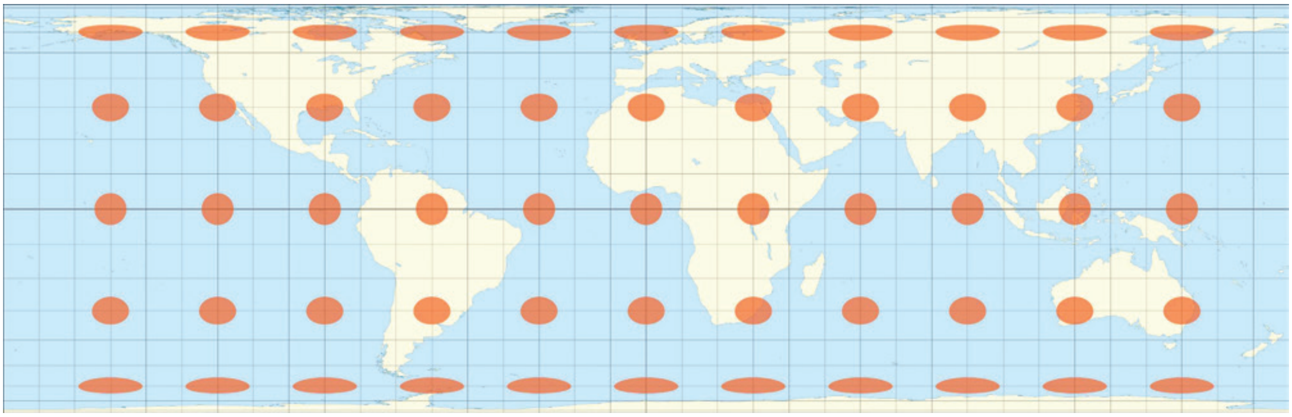

Figure 5: The Lambert equal-area map.

5 Albers and Lambert, One Standard Parallel

In this section, we describe the Albers one standard parallel (equal-area conic projection) and Lambert one standard parallel (conformal conic projection) at latitude ϕ_0 which give good maps around that latitude (cf., [2, 8, 9, 14]).

We start with some geometric properties in a cone tangent to a spherical model of the Earth at latitude ϕ_0.

In Figure 6, ACN and BDN are two meridians separated by a longitude difference of $\Delta\lambda$, and CD is an arc of the circle parallel to the equator. We have $CD = DO'\Delta\lambda$ and $DN' \sin\phi_0 = DO'$ and approximately $\theta \cdot DN' = CD$.

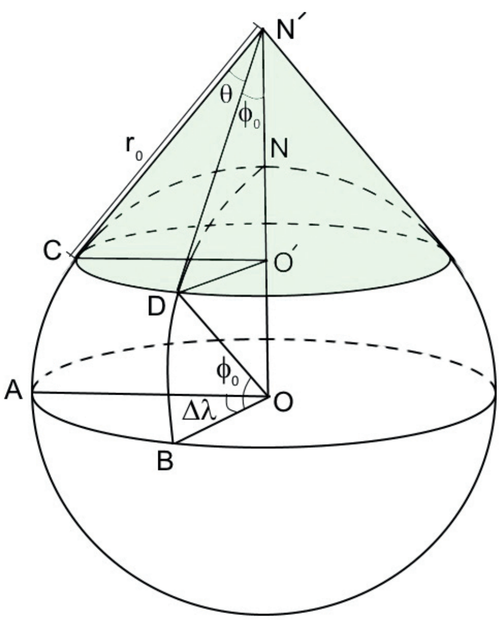

Figure 6: Geometry for angular convergence of the meridians.

Therefore, the first polar coordinate, θ, is a linear function of λ, i. e.,

$$\theta = \Delta\lambda \sin\phi_0. \tag{47}$$

The second polar coordinate, r, is a function of ϕ, i. e.,

$$r = r\left(\phi\right). \tag{48}$$

The constant of the cone, denoted ϱ, is defined from the relation between lengths on the developed cone on the Earth. Let the total angle on the cone, θ_T, corresponding to 2π on the Earth be $\theta_T = d / r_0$, where $d = 2\pi R \cos\phi_0$ is the circumference of the parallel circle to the equator at latitude ϕ_0, and $r_0 = R \cot\phi_0$. Thus $\theta_T = 2\pi \sin\phi_0$, and the constant of the cone is defined as $\varrho = \sin\phi_0$.

Case 1. The Albers projection. Consider a spherical model of the Earth. From Example 1, we know that the first fundamental quantities for the sphere are $e = R^2$ and $g = R^2 \cos^2\phi$ and for a cone (the polar coordinate system) are $E' = 1$ and $G' = r^2$. Hence, from Equations (18) and (19),

$$R^4 \cos^2\phi = r^2 \begin{vmatrix} \dfrac{\partial r}{\partial \phi} & \dfrac{\partial r}{\partial \lambda} \\ \dfrac{\partial \theta}{\partial \phi} & \dfrac{\partial \theta}{\partial \lambda} \end{vmatrix}^2. \tag{49}$$

Using Equations (47) and (48), Equation (49) becomes

$$R^4 \cos^2\phi = r^2 \begin{vmatrix} \dfrac{\partial r}{\partial \phi} & 0 \\ 0 & \sin\phi_0 \end{vmatrix}^2. \tag{50}$$

Solving Equation (50) by knowing the fact that an increase in ϕ corresponds to a decrease in r, one gets

$$r^2 = \frac{-2R^2 \sin\phi}{\sin\phi_0} + c. \tag{51}$$

Imposing the boundary condition $r_0 = R\cot\phi_0$ into Equation (51), $c = 2R^2 + R^2\cot^2\phi_0$, and so after some simplifications, Equation (51) becomes

$$r = \frac{R}{\sin\phi_0}\sqrt{1+\sin^2\phi_0 - 2\sin\phi\sin\phi_0}. \tag{52}$$

The Cartesian plotting equations for a conical projection are defined as follows:

$$x = sr\sin\theta, \qquad y = s\left(r_0 - r\cos\theta\right), \tag{53}$$

where s is the scale factor, θ and r are given respectively by Equations (47) and (52), and $r_0 = R\cot\phi_0$. The origin of the projection has the coordinates λ_0 (the longitude of central meridian) and ϕ_0. Figure 7 shows the Albers projection with one standard parallel. If we let $\phi_0 = 90°$, then Equations (47) and (52) reduce to

$$\theta = \Delta\lambda, \qquad r = R\sqrt{2(1-\sin\phi)}, \tag{54}$$

that are the polar coordinates for the azimuthal equal-area projection, a special case of the Albers projection, see Figure 8. The shape of Tissot's indicatrices in Figures 7 and 8 changes when the latitude changes, but they all have the same area.

Case 2. The Lambert projection. In this case, we consider a spheroidal model of the Earth. From Example 1, the fundamental quantities for this model are $e = M^2$ and $g = N^2\cos^2\phi$, and the fundamental quantities for a cone are $E' = 1$ and $G' = r^2$. Again using Equations (47) and (48), Equation (17) becomes

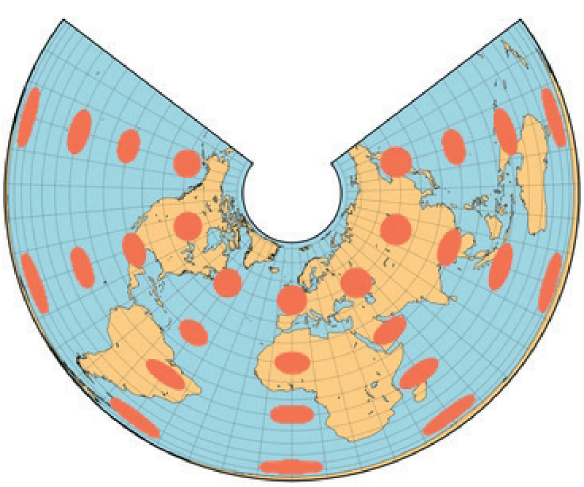

Figure 7: The Albers equal-area map with standard parallel 45°N.

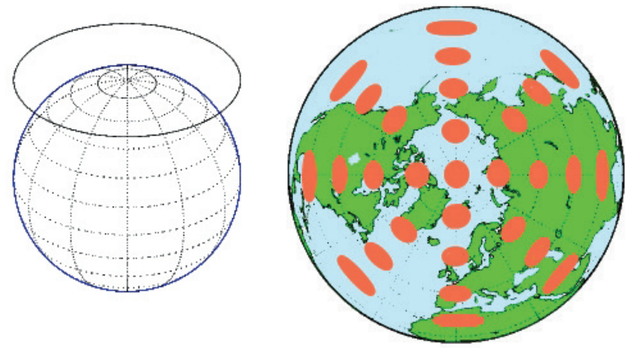

Figure 8: The Albers equal-area azimuthal map.

$$E = \left(\frac{\partial r}{\partial\phi}\right)^2 E' + \left(\frac{\partial\theta}{\partial\phi}\right)^2 G' = \left(\frac{\partial r}{\partial\phi}\right)^2,$$
$$G = \left(\frac{\partial r}{\partial\lambda}\right)^2 E' + \left(\frac{\partial\theta}{\partial\lambda}\right)^2 G' = \sin^2\phi_0 r^2. \tag{55}$$

Substituting these values in Equation (20), integrating, simplifying and noting that r increases as ϕ decreases, one gets

$$r = r_0\left\{\frac{\tan\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\frac{1+e_{ab}\sin\phi}{1-e_{ab}\sin\phi}\right)^{e_{ab}/2}}{\tan\left(\frac{\pi}{4}-\frac{\phi_0}{2}\right)\left(\frac{1+e_{ab}\sin\phi_0}{1-e_{ab}\sin\phi_0}\right)^{e_{ab}/2}}\right\}^{\sin\phi_0}, \tag{56}$$

where

$$r_0 = N_{\phi_0}\cot\phi_0 = \frac{a\cot\phi_0}{\left(1-e_{ab}^2\sin^2\phi_0\right)^{0.5}}. \tag{57}$$

The Cartesian equations are the same as Equation (53) with these new r_0 and r. We show the Lambert projection with one standard parallel in Figure 9. Tissot's indicatrices in this figure have different areas for different latitudes, but they all have the same shape (circle).

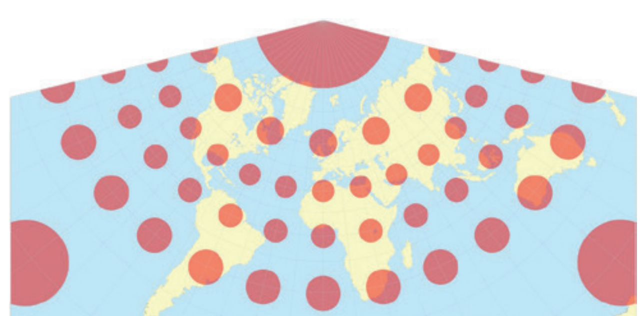

Figure 9: The Lambert conformal map with one standard parallel.

6 Sinusoidal Projection

In this section, we only discuss about the sinusoidal equal-area projection that is a projection of the entire model of the Earth onto a single map, and it gives an adequate whole world coverage [3, 8].

Consider a spherical model of the Earth with the fundamental quantities $e = R^2$ and $g = R^2 \cos^2 \phi$. The first fundamental quantities on a planar mapping surface is $E' = G' = 1$. Substituting these fundamental quantities into Equation (19) (using Equation (18)), one gets

$$R^4 \cos^2 \phi = \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \lambda} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \lambda} \end{vmatrix}^2,$$

which by imposing the conditions $y = R\phi$ and $x = x(\phi, \lambda)$ reduces to

$$R^4 \cos^2 \phi = \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \lambda} \\ R & 0 \end{vmatrix}^2 = R^2 \left(\frac{\partial x}{\partial \lambda} \right)^2. \tag{58}$$

Taking the positive square root of Equation (58) and using the fact that λ and ϕ are independent, one obtains $dx = R \cos \phi d\lambda$, and so by integrating $x = \lambda R \cos \phi + c$. Using the boundary condition $x = 0$ when $\lambda = \lambda_0$, one gets $c = -\lambda_0 R \cos \phi$, and so the plotting equations for the sinusoidal projection become as follow (ϕ and λ in radians):

$$x = sR\Delta\lambda \cos \phi, \qquad y = sR\phi, \tag{59}$$

where s is the scale factor. Figure 10 shows a normalized plot for the sinusoidal projection. In this map, the meridians are sinusoidal curves except the central meridian which is a vertical line and they all meet each other in the poles. This is why this map is known as the sinusoidal map. The x axis is also along the equator. Tissot's indicatrices in Figure 10 change their shapes (the ellipses with different eccentricities indicating angular distortion) toward the poles while having the same areas.

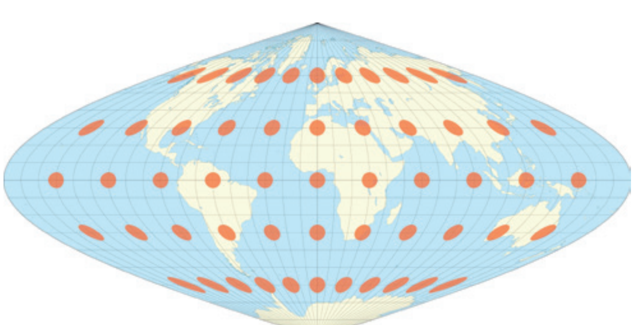

Figure 10: The sinusoidal equal-area projection.

The inverse transformation from the Cartesian to geographic coordinates is simply calculated from Equation (59):

$$\phi = \frac{y}{sR}, \qquad \Delta\lambda = \frac{x}{sR \cos \phi}. \tag{60}$$

7 Some Conventional Projections

In this section, we give the plotting equations for two conventional projections, the simple conic projection (one standard parallel) and the plate carree projection (cf., [3, 8, 13]). As we mentioned earlier, these projections neither preserve the shape nor do they preserve the size, and they are usually used for simple portrayals of the world or regions with minimal geographic data such as index maps.

1. The simple conic projection is a projection that the distances along every meridian are true scale. Suppose that the conic is tangent to the spherical model of the Earth at latitude ϕ_0, see Figure 11. In this figure, we have $r_0 = R \cot \phi_0$. We want to have $DE = DE'$, but $DE' = R(\phi - \phi_0)$. Thus the polar coordinates for this projection are

$$r = r_0 - R(\phi - \phi_0), \qquad \theta = \Delta\lambda \sin \phi_0. \tag{61}$$

Replacing these values into Equation (53) gives its Cartesian coordinates.

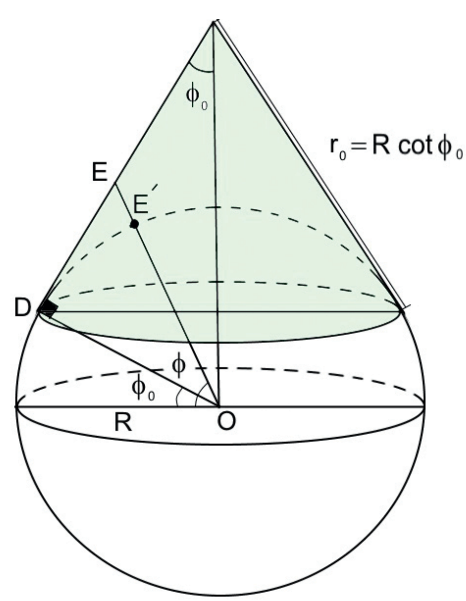

Figure 11: Geometry for the simple conic projection.

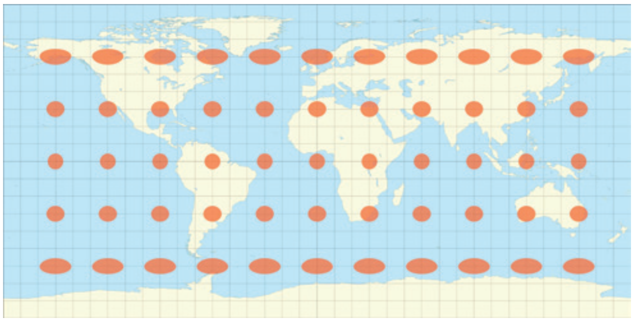

Figure 12: The plate carree map, 10° graticule.

2. The plate carree, the equirectangular projection, is a conventional cylindrical projection that divides the meridians equally the same way as on the sphere. Also, it divides the equator and its parallels equally. The plate carree plotting equations are very simple:

$$x = sR\Delta\lambda, \qquad y = sR\phi, \tag{62}$$

where ϕ and y are in radians. Figure 12 shows the plate carree map with Tissot's indicatrices which are changing their shapes and areas when moving toward the poles indicating that this map is neither equal-area nor conformal.

8 Theory of Distortion

In this section, we discuss about three types of distortions from differential geometry approach: distortions in length, area and angle, and we present them in term of the Gaussian fundamental quantities (cf., [8, 9, 15]).

1. The distortion in length, also known as the scale factor in the surveying and mapping world, is defined as the ratio of a length of a line on a map to the length of the true line on a model of the Earth. More precisely,

$$K_L^2 = \frac{(ds)_M^2}{(ds)_E^2} = \frac{E(d\phi)^2 + G(d\lambda)^2}{e(d\phi)^2 + g(d\lambda)^2}. \tag{63}$$

From Equation (63), the distortion along the meridians ($d\lambda = 0$) is

$$K_m = \sqrt{\frac{E}{e}}, \tag{64}$$

and along the lines parallel to the equator ($d\phi = 0$) is

$$K_e = \sqrt{\frac{G}{g}}. \tag{65}$$

2. The distortion in area is defined as the ratio of an area on a map to the true area on a model of the Earth. From Equation (9) ($f = F = 0$), the area on the map is $A_M = \sqrt{EG}$, and the corresponding area on the model of the Earth is $A_E = \sqrt{eg}$. Thus, the distortion in area is

$$K_A = \frac{A_M}{A_E} = \sqrt{\frac{EG}{eg}} = K_m K_e. \tag{66}$$

In equal-area map projections, from Equation (18), $K_A = K_m K_e = 1$.

3. The distortion in angle is defined as (in percentage):

$$K_\alpha = 100 \cdot \frac{\alpha - \beta}{\alpha}, \tag{67}$$

where α is the angle on a model of the Earth (the azimuth), and β is the projected angle on a map (the azimuth α on the map, cf., Figure 13).

In order to obtain β as a function of the fundamental quantities and α, we first calculate $\sin(\beta \pm \alpha)$. From Figure 13, we have

$$\begin{aligned}\sin(\beta \pm \alpha) &= \sin\beta\cos\alpha \pm \cos\beta\sin\alpha \\ &= \left(\sqrt{G}\cdot\frac{d\lambda}{dS}\right)\left(\sqrt{e}\cdot\frac{d\phi}{ds}\right) \pm \left(\sqrt{E}\cdot\frac{d\phi}{dS}\right)\left(\sqrt{g}\cdot\frac{d\lambda}{ds}\right) \\ &= \left(K_e \pm K_m\right)\frac{d\phi}{dS}\frac{d\lambda}{dS}\sqrt{eg}.\end{aligned} \tag{68}$$

Hence,

$$\sin(\beta - \alpha) = \frac{K_e - K_m}{K_e + K_m}\sin(\beta + \alpha). \tag{69}$$

Define

$$f(\beta) = \sin(\beta - \alpha) - \frac{K_e - K_m}{K_e + K_m}\sin(\beta + \alpha). \tag{70}$$

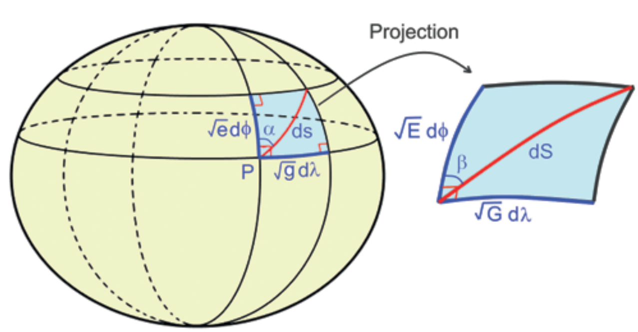

Figure 13: Geometry for differential parallelograms.

Now the goal is to find the roots of f. This can be done by Newton's iteration as follows:

$$\beta_{n+1} = \beta_n - \frac{f\left(\beta_n\right)}{f'\left(\beta_n\right)}, \tag{71}$$

where

$$f'\left(\beta_n\right) = \cos\left(\beta_n - \alpha\right) - \frac{K_e - K_m}{K_e + K_m}\cos\left(\beta_n + \alpha\right). \tag{72}$$

The iteration is rapidly convergent by letting $\beta_0 = \alpha$. In conformal mapping, from Equation (20), $K_e = K_m$, and so the function f will have a unique solution ($\beta = \alpha$).

Example 3 In this example, we show the distortions in length in the Albers projection with one standard parallel. From Example 1, the first fundamental form for the map is

$$\left(ds\right)_M^2 = \left(dr\right)^2 + r^2\left(d\theta\right)^2 = E'\left(dr\right)^2 + G'\left(d\theta\right)^2, \tag{73}$$

and the first fundamental form for the spherical model of the Earth is

$$\left(ds\right)_E^2 = R^2\left(d\phi\right)^2 + R^2\cos^2\phi\left(d\lambda\right)^2 = e\left(d\phi\right)^2 + g\left(d\lambda\right)^2. \tag{74}$$

Taking the derivatives of Equations (47) and (50), one obtains

$$d\theta = \sin\phi_0 d\lambda, \qquad dr = \frac{-R\cos\phi d\phi}{\sqrt{1+\sin^2\phi_0 - 2\sin\phi\sin\phi_0}}, \tag{75}$$

respectively. One may substitute Equation (75) in Equation (73) to get

$$\left(ds\right)_M^2 = \frac{R^2\cos^2\phi}{1+\sin^2\phi_0 - 2\sin\phi\sin\phi_0}\left(d\phi\right)^2 + r^2\sin^2\phi_0\left(d\lambda\right)^2 = E\left(d\phi\right)^2 + G\left(d\lambda\right)^2. \tag{76}$$

Substituting Equations (74) and (76) in Equation (63) gives the total length distortion. Also,

$$K_m = \sqrt{\frac{E}{e}} = \frac{\cos\phi}{\sqrt{1+\sin^2\phi_0 - 2\sin\phi\sin\phi_0}},$$

$$K_e = \sqrt{\frac{G}{g}} = \frac{\sqrt{1+\sin^2\phi_0 - 2\sin\phi\sin\phi_0}}{\cos\phi}, \tag{77}$$

which are functions of ϕ. Clearly, $K_m K_e = 1$.

Example 4 In this example, we first use the first fundamental form to obtain the plotting equations for the Mercator projection, and then we show its length and area distortions. From Example 1, the first fundamental form for the cylindrical surface (the Cartesian coordinate system) is

$$\left(ds\right)_M^2 = \left(dy\right)^2 + \left(dx\right)^2. \tag{78}$$

Taking the derivative of Equation (38) and substituting in Equation (78), one finds

$$\left(ds\right)_M^2 = \left(\frac{dy}{d\phi}\right)^2\left(d\phi\right)^2 + s^2R^2\left(d\lambda\right)^2 = E\left(d\phi\right)^2 + G\left(d\lambda\right)^2, \tag{79}$$

where s is the scale of the map along the equator, $E = (dy/d\phi)^2$ and $G = s^2R^2$. The first fundamental quantities for the spherical model of the Earth are $e = R^2$ and $g = R^2\cos^2\phi$. Substituting these fundamental quantities in Equation (20) and simplifying, one obtains

$$dy = \frac{sRd\phi}{\cos\phi}. \tag{80}$$

It is easy to see that integrating the above differential equation and applying the boundary condition $y(0) = 0$, Equation (43) follows. By Equation (80), $E = (dy/d\phi)^2 = s^2R^2/\cos^2\phi$. Therefore, substituting Equations (74) and (79) in Equation (63), the length distortion will be

$$K_L = \frac{s}{\cos\phi}. \tag{81}$$

It can be seen that $K_L = K_m = K_e$, and so from Equation (66), the distortion in area for the Mercator projection is

$$K_A = K_m K_e = \frac{s^2}{\cos^2\phi}. \tag{82}$$

Hence, in the Mercator projection both length and area distortions are functions of ϕ not λ.

9 Discussions and Conclusions

There are a number of map projections used for different purposes, and we discussed about three major classes of them, equal-area, conformal, and conventional. Users may also create their own map based on their projects by starting with a base map of known projection and scale (e.g., [10]). In certain applications such as global web map visualization, some of the map projections can be also combined to minimize the distortion in both shape and area [1, 4].

In this paper, in cylindrical projections, we assume that the cylinder is tangent to the equator. Making the cylinder tangent to other closed curves on the Earth results good maps in areas close to the tangency. This is also applied for conical and azimuthal projections.

In all projections from a 3-D surface to a 2-D surface, there are distortions in length, shape or size that some of them can be removed (not all) or minimized from the map based on some specific applications. In Section 3, we also showed that projecting a spheroidal model of the Earth to a spherical model of the Earth will distort length, shape and / or angle.

Intelligent map users should have knowledge about the theory of distortion in order to compare and distinguish their maps with the true surface on the Earth that they are studying.

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