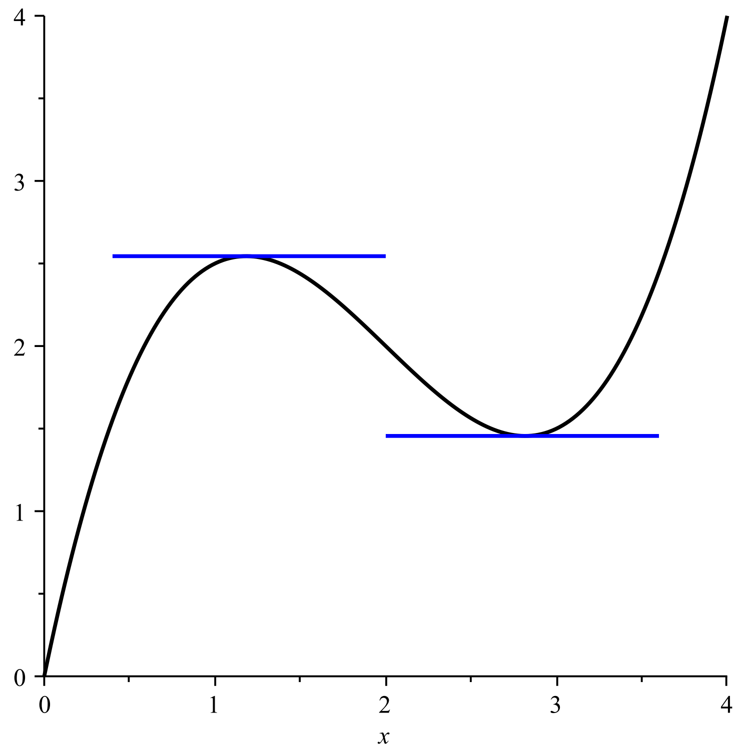
Calculus 3 - Extrema

In calc 1 we used the derivative to determine when a function has a minimum or maximum. We see in this figure below that the extrema are obtained when $f'(x) = 0$.



Furthermore, we can determine min and max by looking at the sign of $f''(x)$. If $f''(x) > 0$ at the critical point then we have a minimum while if $f''(x) < 0$ at the critical point then we have a maximum (the so called second derivative test). So what about functions with more than one independent variable? In the case of two independent variables we also have min's and max's but we also have a third possibility - a saddle.

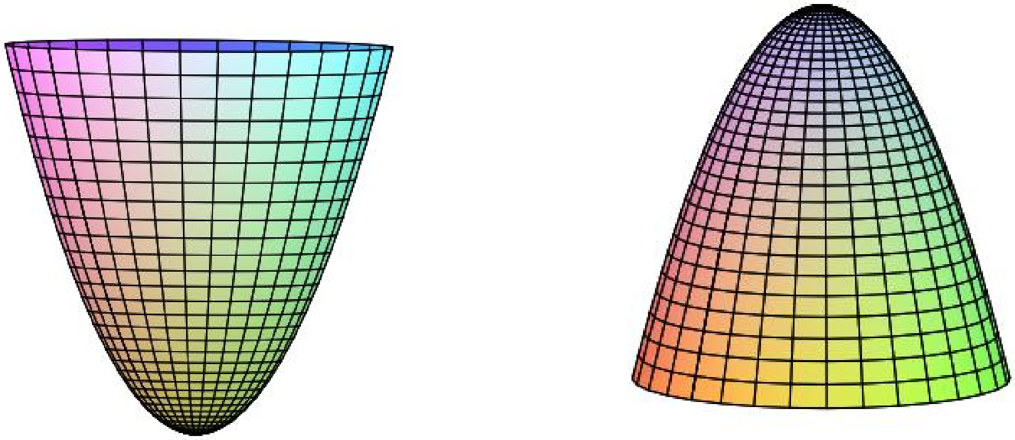


Figure 1: min/max

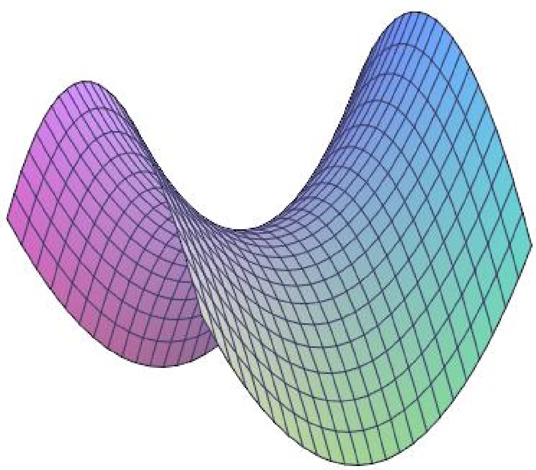


Figure 2: Saddle

So how do we find these? Well, like in calc 1, we define critical points

Def$^{\mathbf{n}}$

Let f be defined on an open region R containing the point (a, b). The point (a, b) is called a critical point if

1. $f_x(a, b) = 0$ and $f_y(a, b) = 0$
2. $f_x(a, b)$ or $f_y(a, b)$ DNE

Example 1. Find the critical points for

$$f(x, y) = x^2 + y^2 - 4x + 2y + 6 \tag{1}$$

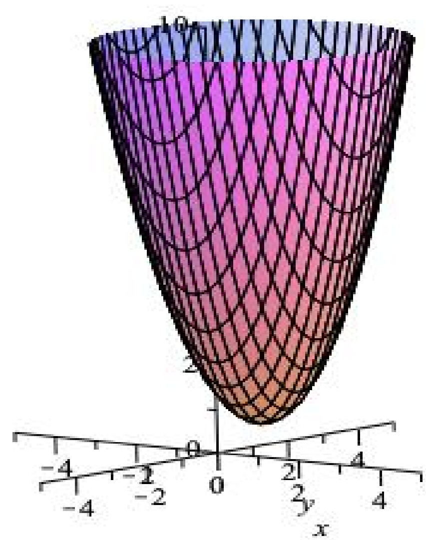
We first calculate the partial derivatives. These are

$$f_x = 2x - 4, \quad f_y = 2y + 2. \tag{2}$$

We then set both these equal to zero so

$$f_x = 2x - 4 = 0, \quad f_y = 2y + 2 = 0. \tag{3}$$

These are easily solved giving $x = 2$ and $y = -1$ so the critical point is $(2, -1)$. The graph of the surface is below and we see we have a minimum.



Example 2. Find the critical points for

$$f(x, y) = x^2 - 2xy - y^2 - 4y. \tag{4}$$

We first calculate the partial derivatives. These are

$$f_x = 2x - 2y, \quad f_y = -2x - 2y - 4. \tag{5}$$

We then set both these equal to zero so

$$f_x = 2x - 2y = 0, \quad f_y = -2x - 2y - 4 = 0. \tag{6}$$

Solving the first gives $y = x$ and substituting this into the second gives $-4x - 4 = 0$ so $x = -1$ and $y = -1$ so the critical point is $(-1, -1)$.
Example 3. Find the critical points for

$$f(x, y) = x^2y + xy^2 - 3xy. \tag{7}$$

We first calculate the partial derivatives. These are

$$f_x = 2xy + y^2 - 3y, \quad f_y = x^2 + 2xy - 3x. \tag{8}$$

We then set both these equal to zero so

$$f_x = 2xy + y^2 - 3y = 0, \quad f_y = x^2 + 2xy - 3x = 0, \tag{9}$$

or

$$y(2x + y - 3) = 0, \quad x(x + 2y - 3) = 0. \tag{10}$$

So now we have 4 possibilities

$$(0,0), \quad (0,3), \quad (3,0) \quad (1,1)$$

So how do we know if we have min's, max's or saddles. We have a second derivative test

Def$^{\mathbf{n}}$

Let f have continuous partial derivatives and

$$f_x(a,b) = 0 \quad \text{and} \quad f_y(a,b) = 0. \tag{11}$$

Define Δ as

$$\Delta = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}^2(a,b). \tag{12}$$

1. If $\Delta > 0$ and $f_{xx}(a,b) > 0$ then f has a relative minimum at (a,b)
2. If $\Delta > 0$ and $f_{xx}(a,b) < 0$ then f has a relative maximum at (a,b)
3. If $\Delta < 0$ then f has a saddle at (a,b)
4. If $\Delta = 0$ test fails

Let us re-examine the previous examples.

Example 1.

The critical point was $(2,-1)$. We calculate the second partial derivatives so

$$f_{xx} = 2, \;\; f_{xy} = 0, \;\; f_{yy} = 2. \tag{13}$$

In this case $\Delta = 4 > 0$ and $f_{xx} > 0$ so from our test, $(2-1)$ is a relative minimum.

Example 2.

The critical point was $(-1,-1)$. We calculate the second partial derivatives so

$$f_{xx} = 2, \;\; f_{xy} = -2, \;\; f_{yy} = -2. \tag{14}$$

In this case $\Delta = (2)(-2) - (-2)^2 = -8 < 0$ so from our test we have a saddle.

Example 3. Here we found the critical points to be

$$(0,0), \quad (0,3), \quad (3,0) \quad (1,1).$$

We calculate the second partial derivatives to find

$$f_{xx} = 2y, \quad f_{xy} = 2x + 2y - 3, \;\; f_{yy} = 2x, \tag{15}$$

so

$$\Delta = 4xy - (2x + 2y - 3)^2 \tag{16}$$

So now we test the critical points.

CP	Δ	f_{xx}	type
$(0,0)$	-9	NA	saddle
$(0,3)$	-9	NA	saddle
$(3,0)$	-9	NA	saddle
$(1,1)$	1	> 0	min

Extrema on Finite Domains

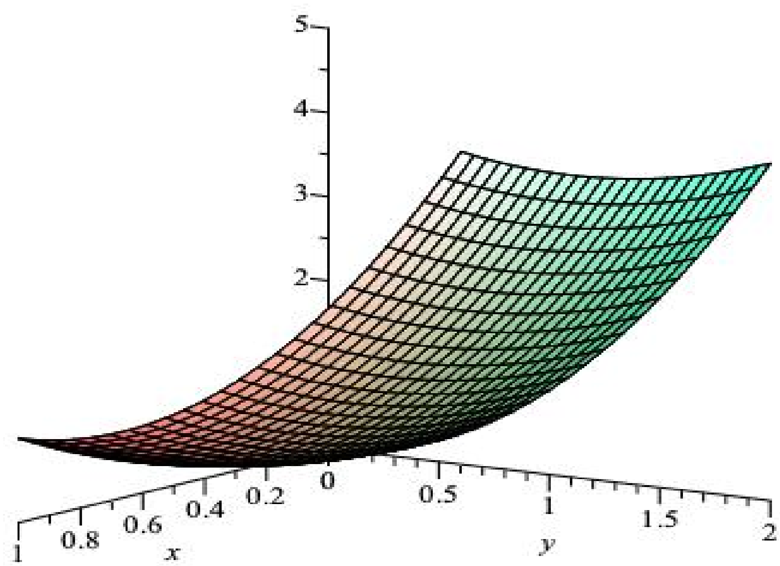
We know from Calc 1 that every continuous function on a closed interval will have a absolute minimum and maximum. It will be inside the interval or on the boundary. Now we consider functions of two variables $z = f(x, y)$ on a finite domain. For example

$$f(x, y) = x^2 + y^2 \tag{17}$$

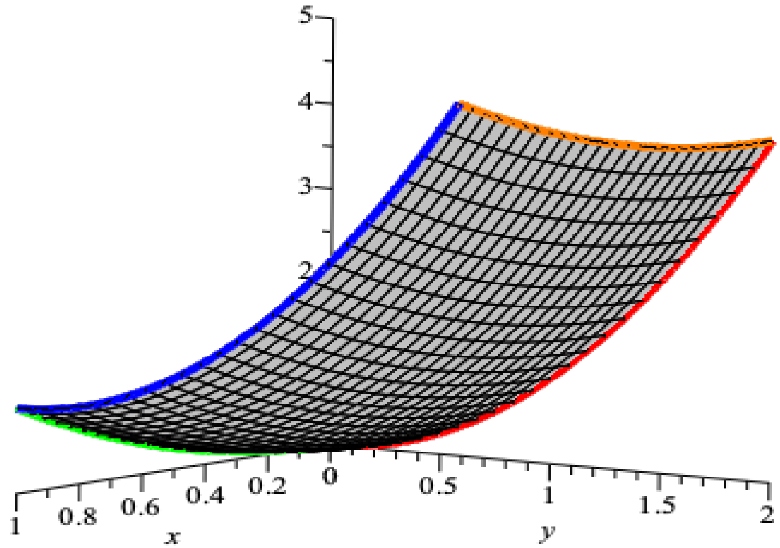
on the domain

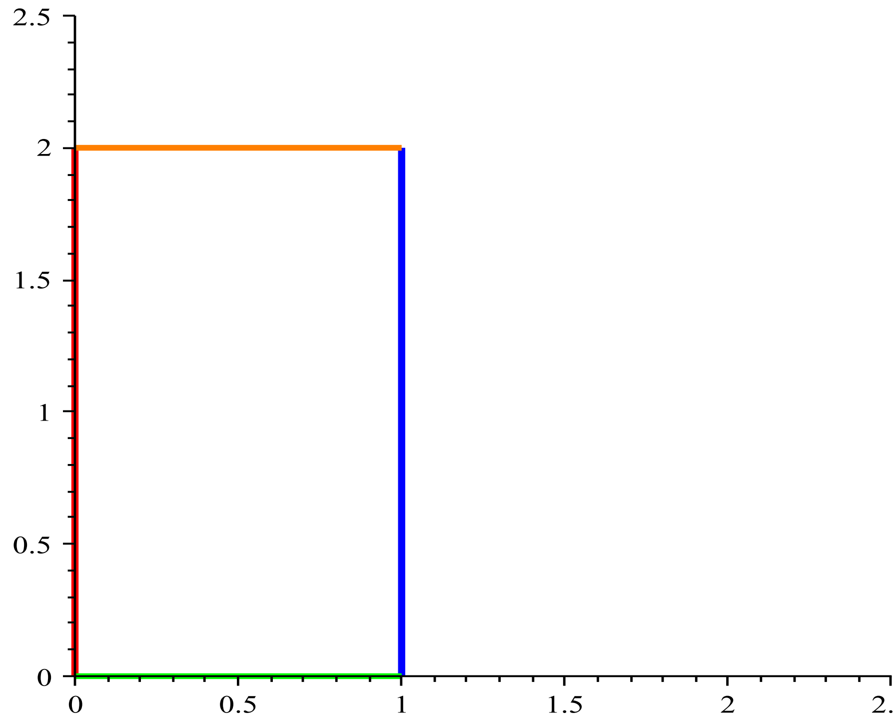
$$0 \leq x \leq 1, \quad 0 \leq y \leq 2. \tag{18}$$

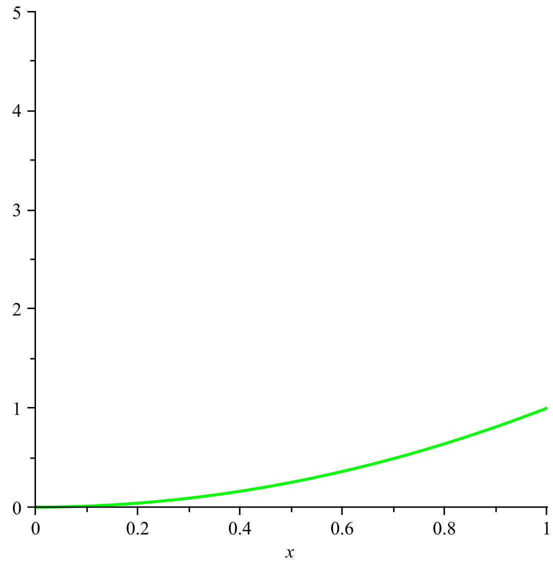
we see in the graph there is a maximum and minimum. So where do we look? Either at an interior point of the domain or on the domain itself. The



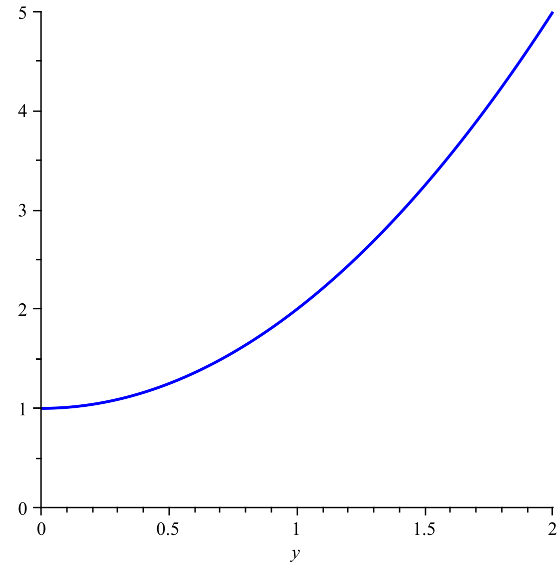
domain is in the graph below


5
4
3
2
1
0.8
0.6
0.4
0.2
0
0.5
1
1.5
2
x
y

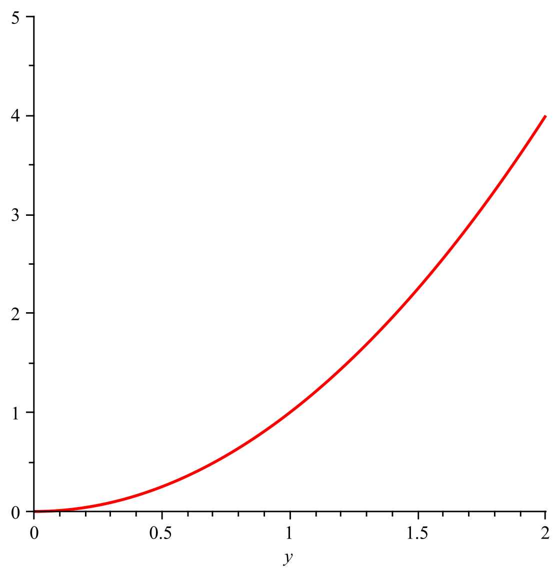

2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5



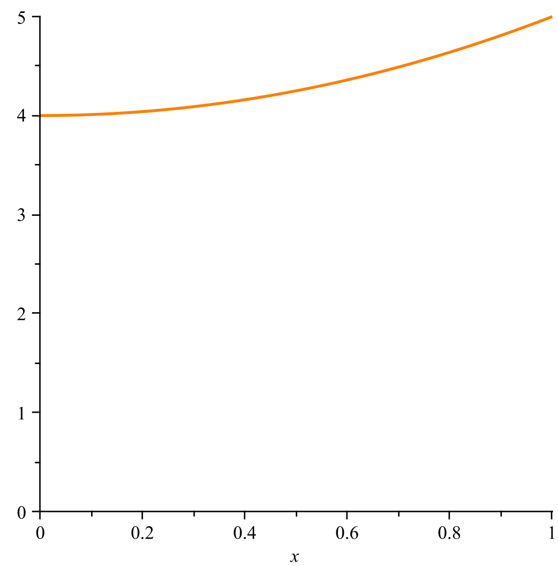
$z = x^2$



$z = 1 + y^2$



$z = y^2$



$z = 4 + x^2$

and we can see that the minimum occurs at $(0, 0)$ and the maximum at $(1, 2)$