International Journal of Modern Science and Technology *Vol. 2, No. 4, 2017. Page 158-167.*
http://www.ijmst.co/ ***ISSN: 2456-0235.***

Research Article

Oscillation of Even order Impulsive Neutral Partial Differential Equations with Distributed Delay and Damping

V. Sadhasivam, **T. Kalaimani***, **T. Raja**

PG and Research Department of Mathematics, Thiruvalluvar Government Arts College, Rasipuram - 637 401. India.

*Corresponding author's e-mail: kalaimaths4@gmail.com

Abstract

The present work considered a class of boundary value problems associated with even order impulsive neutral partial functional differential equations with continuous distributed deviating arguments and damping term. Necessary and Sufficient conditions are obtained for the oscillation of solutions using impulsive differential inequalities and integral averaging scheme with Robin boundary condition. Examples are specified to illustrate the important results.

Keywords: Neutral partial differential equations; Oscillation; Impulse; Distributed deviating arguments.

Introduction

The oscillation theory of ordinary differential equations marks its establishment in an explore article of Sturm [1] in 1836 and for partial differential equations through Hartman and Wintner [2] in 1955. In 1989, the initial work on impulsive delay differential equations [3] was in print and their substances were included in monograph [4]. In addition the most important effort concluded in [5] on impulsive partial differential equations in 1991. Numerous substantial phenomena are expressed in terms of second order equations. The theoretical background of the second and higher order equations is common and for this reason, we study the higher order equations. The spacious interest on qualitative studies of ordinary and partial functional differential equations is came back to their varieties of applications in various fields of science and machinery [6-10].

The oscillation of impulsive and non-impulsive parabolic and hyperbolic equations has been widely studied in the literature, we refer the readers to the papers [11-18] and the references they are cited. Curiously the minority significant consequences on higher order partial differential equations with continuous distributed deviating arguments have been studied in [19-23]. But these are not considered with impulse effect. Consequently, it is necessary to study with impulse effect on the oscillation of higher order partial differential equations. To the best of authors' acquaintance, there are no scientific articles on the oscillation of higher order impulsive neutral partial differential equations with continuous distributed deviating arguments and damping term. In this fashion, we initiate oscillatory results for even order impulsive neutral partial differential equations with continuous distributed deviating arguments and damping. Focal results of this manuscript expand and improve numerous findings in the earlier publications of non-impulsive type equations. We think likely that this primary work achieve the absorption of numerous researchers working on the even order impulsive partial functional differential equations. In the current study will the follow even order impulsive neutral partial functional differential equation with continuous distributed deviating arguments and damping.

$$\left.\begin{aligned}&\frac{\partial}{\partial t}\left[r(t)\frac{\partial^{m-1}}{\partial t^{m-1}}\big(u(x,t)+c(t)u(x,\tau(t))\big)\right]+p(t)\frac{\partial^{m-1}}{\partial t^{m-1}}\big(u(x,t)+c(t)u(x,\tau(t))\big)\\&+\int_a^b q(t,\xi)u(x,\sigma(t,\xi))d\eta(\xi)=a(t)\Delta u(x,t)+\int_a^b b(t,\xi)\Delta u(x,\rho(t,\xi))d\eta(\xi),\\&\qquad\qquad\qquad\qquad t\neq t_k,\ (x,t)\in\Omega\times(0,+\infty)\equiv G\\&\frac{\partial^{(i)}u(x,t_k^+)}{\partial t^{(i)}}=I_k^{(i)}\left(x,t_k,\frac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}\right),\qquad t=t_k,k=1,2,\cdots,\qquad i=0,1,2,\cdots,m-1\end{aligned}\right\}\quad(1)$$

Received: 12.04.2017; Received after Revision: 20.04.2017; Accepted: 22.04.2017; Published: 25.04.2017

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$.
Equation (1) is enhancement with the following Robin boundary condition 2.

$$\alpha(x)\frac{\partial u(x,t)}{\partial \gamma}+\beta(x)u(x,t)=0,$$
$$(x,t)\in\partial\Omega\times(0,+\infty) \quad (2)$$

where γ is the outer surface normal vector to $\partial\Omega$ and $\alpha,\beta\in C(\partial\Omega,\mathbb{R}^+)$, $\alpha^2(x)+\beta^2(x)\neq 0$.

In the sequel, we assume that the following hypotheses (H) hold:

(H_1) $r(t)\in C'([0,+\infty),(0,+\infty))$, $r'(t)\geq 0$, $p(t)\in C([0,+\infty),\mathbb{R})$, $\int_{t_0}^{+\infty}\frac{1}{R(s)}ds=+\infty$, where $R(t)=\exp\left(\int_{t_0}^{t}\frac{r'(s)+p(s)}{r(s)}ds\right)$.

(H_2) $c(t)\in C^m([0,+\infty),[0,+\infty))$, $a(t)\in PC([0,+\infty),[0,+\infty))$, where PC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t=t_k$, $k=1,2,\cdots$, and left continuous at $t=t_k$, $k=1,2,\cdots$, $\tau(t)\in C([0,+\infty),\mathbb{R})$, $\lim_{t\to+\infty}\tau(t)=+\infty$, $q(t,\xi)\in C([0,+\infty)\times[a,b],[0,+\infty))$.

(H_3) $b(t,\xi)\in C([0,+\infty)\times[a,b],[0,+\infty))$, $\sigma(t,\xi),\rho(t,\xi)\in C([0,+\infty)\times[a,b],\mathbb{R})$, $\sigma(t,\xi)\leq t$, $\rho(t,\xi)\leq t$ for $\xi\in[a,b]$, $\sigma(t,\xi)$ and $\rho(t,\xi)$ are nondecreasing with respect to t and ξ respectively and $\liminf_{t\to+\infty,\xi\in[a,b]}\sigma(t,\xi)=\liminf_{t\to+\infty,\xi\in[a,b]}\rho(t,\xi)=+\infty$.

(H_4) There exist a function $\theta(t)\in C([0,+\infty),[0,+\infty))$ satisfying $\theta(t)\leq\sigma(t,a)$, $\theta'(t)>0$ and $\lim_{t\to+\infty}\theta(t)=+\infty$, $\eta(\xi):[a,b]\to\mathbb{R}$ is nondecreasing and the integral is a Stieltjes integral in (1).

(H_5) $\frac{\partial^{(i)}u(x,t)}{\partial t^{(i)}}$ are piecewise continuous in t with discontinuities of first kind only at $t=t_k$, $k=1,2,\cdots$, and left continuous at $t=t_k$, $\frac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}=\frac{\partial^{(i)}u(x,t_k^-)}{\partial t^{(i)}}$, $k=1,2,\cdots$, $i=0,1,2,\cdots,m-1$.

(H_6) $I_k^{(i)}\left(x,t_k,\frac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}\right)\in PC(\bar{\Omega}\times[0,+\infty)\times\mathbb{R},\mathbb{R})$, $k=1,2,\cdots$, $i=0,1,2,\cdots,m-1$, and there exist positive constants $a_k^{(i)}, b_k^{(i)}$ with $b_k^{(m-1)}\leq a_k^{(0)}$ such that for $i=0,1,2,\cdots,m-1$, $k=1,2,\cdots$,

$$a_k^{(i)}\leq\frac{I_k^{(i)}\left(x,t_k,\frac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}\right)}{\frac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}}\leq b_k^{(i)}.$$

The present paper is organized as follows: In section 2, we present the definitions and notations will be needed. In section 3, we deal with the oscillation of the problem (1) and (2). Section 4, presents examples to illustrate the main results.

Preliminaries

In the preliminaries section, we begin with definitions and known results which are required throughout this paper.

Definition 2.1. A solution u of the problem (1) is a function $u\in C^m(\bar{\Omega}\times[t_{-1},+\infty),\mathbb{R})\cap C(\bar{\Omega}\times[\hat{t}_{-1},+\infty),\mathbb{R})$ *that satisfies (1), where*

$$t_{-1}:=\min\left\{0,\inf_{t\geq 0}\tau(t)\right\} \quad and$$
$$\hat{t}_{-1}:=\min\left\{0,\min_{\xi\in[a,b]}\left\{\inf_{t\geq 0}\sigma(t,\xi)\right\},\min_{\xi\in[a,b]}\left\{\inf_{t\geq 0}\rho(t,\xi)\right\}\right\}.$$

Definition 2.2. The solution u of the problem $(1),(2)$ is said to be oscillatory in the domain G if for any positive number ℓ there exist a point $(x_0,t_0)\in\Omega\times[\ell,+\infty)$ such that $u(x_0,t_0)=0$ holds.

Definition 2.3. A function $V(t)$ is said to be eventually positive (negative) if there exists a $t_1\geq t_0$ such that $V(t)>0$ (<0) holds for all $t\geq t_1$.

Lemma 2.1. [24] Suppose that the smallest eigenvalue $\lambda_0>0$ of the eigenvalue problem

$$\left.\begin{aligned}\Delta\omega(x)+\lambda\omega(x)&=0 \quad in \quad \Omega\\ \omega(x)&=0 \quad on \quad \partial\Omega\end{aligned}\right\} \quad (3)$$

and $\Phi(x)>0$ is the corresponding eigenfunction of λ_0. Then $\lambda_0=0$, $\Phi(x)=1$ as $\beta=0$ $(x\in\Omega)$ and $\lambda_0>0$, $\Phi(x)>0$ $(x\in\Omega)$ as $\beta(x)\not\equiv 0$ $(x\in\partial\Omega)$.

Lemma 2.2. [25] Let $y(t)$ be a positive and n times differentiable function on $[0,+\infty)$. If $y^{(n)}(t)$ is constant sign and not identically zero on any ray $[t_1,+\infty)$ for $t_1>0$, then there exists a $t_y\geq t_1$ and integer l $(0\leq l\leq n)$, with $n+l$ even for $y(t)y^{(n)}(t)\geq 0$ or $n+l$ odd for $y(t)y^{(n)}(t)\leq 0$; and for $t\geq t_y$, $y(t)y^{(k)}(t)>0$, $0\leq k\leq l$; $(-1)^{k-l}y(t)y^{(k)}(t)>0$, $l\leq k\leq n$.

Lemma 2.3. [26] Suppose that the conditions of Lemma 2.2 is satisfied, and

$$y^{(n-1)}(t)y^{(n)}(t) \le 0, \quad t \ge t_y.$$

Then there exist constant $\mu \in (0,1)$ and $M > 0$ such that for sufficiently large t

$$|y'(\mu t)| \ge M t^{n-2} |y^{(n-1)}(t)|.$$

Lemma 2.4. [27] If X and Y are nonnegative, then

$$X^\mu - \mu X Y^{\mu-1} + (\mu - 1)Y^\mu \ge 0, \quad \mu > 1$$
$$X^\mu - \mu X Y^{\mu-1} - (1-\mu)Y^\mu \le 0, \quad 0 < \mu < 1,$$

where the equality holds if and only if $X = Y$.

For each positive solution $u(x,t)$ of the problem (1), (2) we combine the functions given below

$$V(t) = \int_\Omega u(x,t)\Phi(x)dx, \qquad F(t) = \frac{\varphi'(t)}{\varphi(t)} - \frac{\varphi(t)p(t)}{r(t)}$$

$$L(t) = \frac{M(\theta(t))^{m-2}\theta'(t)}{r(t)} \quad and \quad G(t) = g_0 \int_a^b q(t,\xi)d\eta(\xi)$$

where $g_0 = 1 - c(\sigma(t,\xi))$.

Results and discussion

Establish necessary and sufficient conditions for the oscillation of all solutions of the problem (1), (2).

Theorem 3.1. The boundary value problem $(1),(2)$ is oscillatory in G iff all solutions of the impulsive differential equation

$$\frac{d}{dt}\left[r(t)\frac{d^{m-1}}{dt^{m-1}}(V(t) + c(t)V(\tau(t)))\right] + p(t)\frac{d^{m-1}}{dt^{m-1}}(V(t) + c(t)V(\tau(t)))$$
$$+ \int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi) + \lambda_0 a(t)V(t)$$
$$+\lambda_0 \int_a^b b(t,\xi)V(\rho(t,\xi))d\eta(\xi) = 0, \quad t \ne t_k$$
$$a_k^{(i)} \le \frac{\frac{\partial^{(i)}V(t_k^+)}{\partial t^{(i)}}}{\frac{\partial^{(i)}V(t_k)}{\partial t^{(i)}}} \le b_k^{(i)}, \quad t = t_k, \quad k = 1,2,\cdots, \quad i = 0,1,2,\cdots,m-1$$

.........(4)

are oscillatory, when $\beta(x) \not\equiv 0$ for $x \in \partial\Omega$ and λ_0 is the smallest eigenvalue of (3).

Proof. (i) Sufficient part: Assume that there exist a nonoscillatory solution $u(x,t)$ of the boundary value problem $(1),(2)$ and $u(x,t) > 0$. By the hypothesis (H_3), that there exist a $t_1 > t_0 > 0$ such that $\tau(t) \ge t_0$, $\sigma(t,\xi) \ge t_0$, $\rho(t,\xi) \ge t_0$ for $(t,\xi) \in [t_1, +\infty) \times [a,b]$, we get that

$u(x,\tau(t)) > 0 \quad for \quad (x,t) \in \Omega \times [t_1, +\infty),$

$u(x,\sigma(t,\xi)) > 0 \quad for \quad (x,t,\xi) \in \Omega \times [t_1,+\infty) \times [a,b]$

$and \quad u(x,\rho(t,\xi)) > 0 \quad for \quad (x,t) \in \Omega \times [t_1,+\infty) \times [a,b].$

Multiplying both sides of equation (1) by $\Phi(x) > 0$ and integrating with respect to x over the domain Ω, we attain

$$\left.\begin{aligned}
&\frac{d}{dt}\left[r(t)\frac{d^{m-1}}{dt^{m-1}}\left(\int_\Omega u(x,t)\Phi(x)dx + \int_\Omega c(t)u(x,\tau(t))\Phi(x)dx\right)\right]\\
&+p(t)\frac{d^{m-1}}{dt^{m-1}}\left(\int_\Omega u(x,t)\Phi(x)dx + \int_\Omega c(t)u(x,\tau(t))\Phi(x)dx\right)\\
&+\int_\Omega \int_a^b q(t,\xi)u(x,\sigma(t,\xi))\Phi(x)d\eta(\xi)dx\\
&= a(t)\int_\Omega \Delta u(x,t)\Phi(x)dx + \int_\Omega \int_a^b b(t,\xi)\Delta u(x,\rho(t,\xi))\Phi(x)d\eta(\xi)dx.
\end{aligned}\right\}$$

......(5)

From Green's formula and boundary condition (2), we see that

$$\int_\Omega \Delta u(x,t)\Phi(x)dx = \int_{\partial\Omega}\left[\Phi(x)\frac{\partial u(x,t)}{\partial\gamma} - u(x,t)\frac{\partial\Phi(x)}{\partial\gamma}\right]dS + \int_\Omega u(x,t)\Delta\Phi(x)dx$$
$$= \int_{\partial\Omega}\left[\Phi(x)\frac{\partial u(x,t)}{\partial\gamma} - u(x,t)\frac{\partial\Phi(x)}{\partial\gamma}\right]dS - \lambda_0 \int_\Omega u(x,t)\Phi(x)dx, \quad t \ge t_1.$$

Where dS is surface component on $\partial\Omega$. When $\alpha(x) \equiv 0$, $x \in \partial\Omega$, then by (1) we get $\beta(x) \not\equiv 0$, $u(x,t) = 0$, $(x,t) \in \partial\Omega \times \mathbb{R}^+$, we have

$$\int_{\partial\Omega}\left(\Phi(x)\frac{\partial u(x,t)}{\partial\gamma} - u(x,t)\frac{\partial\Phi(x)}{\partial\Omega}\right)dS \equiv 0, \quad t \ge t_1, \quad t \ne t_k.$$

If $\alpha(x) \not\equiv 0, \alpha^2(x) + \beta^2(x) \ne 0$, where $\alpha, \beta \in C(\partial\Omega, \mathbb{R}^+)$ and $\partial\Omega$ is piecewise smooth, without loss of generality, we can assume that $\alpha(x) > 0$, $x \in \partial\Omega$. Then by (1) and (3), we get that

$$\int_{\partial\Omega}\left(\Phi(x)\frac{\partial u(x,t)}{\partial\gamma} - u(x,t)\frac{\partial\Phi(x)}{\partial\gamma}\right)dS$$
$$= \int_{\partial\Omega}\left(-\Phi(x)\frac{\beta(x)}{\alpha(x)}u(x,t) + \frac{\beta(x)}{\alpha(x)}\Phi(x)u(x,t)\right)dS = 0, \quad t \ge t_1.$$

Then by Lemma 2.1, we have

$$\left.\begin{aligned}
&\int_\Omega \Delta u(x,t)\Phi(x)dx = -\lambda_0 \int_\Omega u(x,t)\Phi(x)dx, \quad t \ge t_1\\
&= -\lambda_0 V(t) \qquad ....(6)\\
&\text{and}\\
&\int_\Omega \Delta u(x,\rho(t,\xi))\Phi(x)dx = -\lambda_0 \int_\Omega u(x,\rho(t,\xi))\Phi(x)dx, \quad t \ge t_1\\
&= -\lambda_0 V(\rho(t,\xi)). \qquad .....(7)
\end{aligned}\right.$$

It is easy to see that

$$\int_\Omega \int_a^b q(t,\xi)u(x,\sigma(t,\xi))\Phi(x)d\eta(\xi)dx$$
$$= \int_a^b q(t,\xi)\int_\Omega u(x,\sigma(t,\xi))\Phi(x)dx d\eta(\xi)$$
$$= \int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi) \text{.....(8)}$$

Combining (5) - (8), we get that

$$\left.\begin{aligned}
&\frac{d}{dt}\left[r(t)\frac{d^{m-1}}{dt^{m-1}}(V(t) + c(t)V(\tau(t)))\right]\\
&+p(t)\frac{d^{m-1}}{dt^{m-1}}(V(t) + c(t)V(\tau(t)))\\
&+\int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi) + \lambda_0 a(t)V(t)\\
&+\lambda_0 \int_a^b b(t,\xi)V(\rho(t,\xi)) = 0, \quad t \ge t_1, \quad t \ne t_k.
\end{aligned}\right\} \text{...(9)}$$

Multiplying both sides of the equation (1) by $\Phi(x) > 0$, integrating with respect to x over the domain Ω, and from (H_6), we obtain

$$a_k^{(i)} \le \frac{\frac{\partial^{(i)} u(x,t_k^+)}{\partial t^{(i)}}}{\frac{\partial^{(i)} u(x,t_k)}{\partial t^{(i)}}} \le b_k^{(i)}.$$

According to $V(t) = K_\Phi \int_\Omega u(x,t)\Phi(x)dx$, we have

$$a_k^{(i)} \le \frac{\frac{\partial^{(i)} V(x,t_k^+)}{\partial t^{(i)}}}{\frac{\partial^{(i)} V(x,t_k)}{\partial t^{(i)}}} \le b_k^{(i)}.$$

Therefore $V(t)$ is an eventually positive solution of (4), which contradicts the fact that all solutions of equation (4) are oscillatory.

(ii) Necessary part: Suppose that equation (4) has a nonoscillatory solution $\tilde{V}(t) > 0$. Without loss of generality we assume $\tilde{V}(t) > 0$ for $t \ge t_* \ge 0$, where t_* is some large number. From (4), we have

$$\begin{aligned}&\frac{d}{dt}\left[r(t)\frac{d^{m-1}}{dt^{m-1}}\left(\tilde{V}(t) + c(t)\tilde{V}(\tau(t))\right)\right]\\ &+p(t)\frac{d^{m-1}}{dt^{m-1}}\left(\tilde{V}(t) + c(t)\tilde{V}(\tau(t))\right)\\ &+\int_a^b q(t,\xi)\tilde{V}(\sigma(t,\xi))d\eta(\xi) + \lambda_0 a(t)\tilde{V}(t)\\ &+\lambda_0\int_a^b b(t,\xi)\tilde{V}(\rho(t,\xi))d\eta(\xi) = 0, \quad t \ge t_*,\ t \ne t_k,\ x \in \Omega\end{aligned} \quad \text{.....(10)}$$

Multiplying both sides of (10) by $\Phi(x) > 0$ we obtain

$$\left.\begin{aligned}&\frac{\partial}{\partial t}\left[r(t)\frac{\partial^{m-1}}{\partial t^{m-1}}\left(\tilde{V}(t)\Phi(x) + c(t)\tilde{V}(\tau(t))\Phi(x)\right)\right]\\ &+p(t)\frac{\partial^{m-1}}{\partial t^{m-1}}\left(\tilde{V}(t)\Phi(x) + c(t)\tilde{V}(\tau(t))\Phi(x)\right)\\ &+\int_a^b q(t,\xi)\tilde{V}(\sigma(t,\xi))\Phi(x)d\eta(\xi) + \lambda_0 a(t)\tilde{V}(t)\Phi(x)\\ &+\lambda_0\int_a^b \Delta\tilde{V}(\rho(t,\xi))\Phi(x)d\eta(\xi) = 0, \quad t \ge t_*,\ x \in \Omega.\end{aligned}\right\} \text{..(11)}$$

Let $\tilde{u}(x,t) = \tilde{V}(t)\Phi(x)$, $(x,t) \in \Omega \times [0,+\infty)$. By Lemma 2.1, we have $\Delta w(x) = -\lambda_0 w(x)$, $x \in \Omega$. Then (11) gives

$$\begin{aligned}&\frac{\partial}{\partial t}\left[r(t)\frac{\partial^{m-1}}{\partial t^{m-1}}\left(\tilde{u}(x,t) + c(t)\tilde{u}(x,\tau(t))\right)\right]\\ &+p(t)\frac{\partial^{m-1}}{\partial t^{m-1}}\left(\tilde{u}(x,t) + c(t)\tilde{u}(x,\tau(t))\right)\\ &+\int_a^b q(t,\xi)\tilde{u}(x,\sigma(t,\xi))\Phi(x)d\eta(\xi)\\ &= a(t)\Delta\tilde{u}(x,t) + \int_a^b b(t,\xi)\Delta\tilde{u}(x,\rho(t,\xi))d\eta(\xi), \quad t \ge t_*,\ x \in \Omega.\end{aligned} \quad \text{.....(12)}$$

Multiplying both sides of equation (10) by $\Phi(x) > 0$, we have

$$a_k^{(i)}\frac{\partial^{(i)}}{\partial t^{(i)}}\tilde{V}(t_k)\Phi(x) \le \frac{\partial^{(i)}}{\partial t^{(i)}}\tilde{V}(t_k^+)\Phi(x) \le b_k^{(i)}\frac{\partial^{(i)}}{\partial t^{(i)}}\tilde{V}(t_k)\Phi(x) \quad \text{.....(13)}$$

Since

$\tilde{u}(x,t) = \tilde{V}(t)\Phi(x)$, $(x,t) \in \Omega \times [0,+\infty)$,

$$a_k^{(i)}\frac{\partial^{(i)}}{\partial t^{(i)}}\tilde{u}(x,t_k) \le \frac{\partial^{(i)}}{\partial t^{(i)}}\tilde{u}(x,t_k^+) \le b_k^{(i)}\frac{\partial^{(i)}}{\partial t^{(i)}}\tilde{u}(x,t_k)$$

$$\frac{\partial^{(i)}}{\partial t^{(i)}}\tilde{u}(x,t_k^+) = I_k^{(i)}\left(x,t_k,\frac{\partial^{(i)}}{\partial t^{(i)}}\tilde{u}(x,t_k)\right).$$

Which shows that

$\tilde{u}(x,t) = \tilde{V}(t)\Phi(x)$, $(x,t) \in \Omega \times [t_*,+\infty)$,

satisfies (1). From Lemma 2.1, we get that

$$\alpha(x)\frac{\partial w(x)}{\partial\gamma} + \beta(x)w(x) = 0, \quad x \in \partial\Omega$$

which implies that

$$\alpha(x)\frac{\partial\tilde{u}(x,t)}{\partial\gamma} + \beta(x)\tilde{u}(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+. \quad \text{.......(14)}$$

Hence $\tilde{u}(x,t) = \tilde{V}(t)\Phi(x) > 0$ is a nonoscillatory solution of the problem (1),(2) which is a contradiction.

Remark 3.1. Theorem 3.1 shows that the oscillation of problem (1), (2) is equivalent to the oscillation of the impulsive differential equation (4).

Theorem 3.2. If $\beta(x) \not\equiv 0$ for $x \in \partial\Omega$ and the impulsive differential inequality

$$\left\{\begin{aligned}&\left(r(t)Z^{(m-1)}(t)\right)' + p(t)Z^{(m-1)}(t) + g_0\int_a^b q(t,\xi)Z(\theta(t))d\eta(\xi) \le 0\\ &a_k^{(i)} \le \frac{\frac{\partial^{(i)}Z(t_k^+)}{\partial t^{(i)}}}{\frac{\partial^{(i)}Z(t_k)}{\partial t^{(i)}}} \le b_k^{(i)}, \quad t = t_k, \quad k = 1,2,\cdots, \quad i = 0,1,2,\cdots,m-1\end{aligned}\right\} \quad \text{.......(15)}$$

has no eventually positive solutions, then every solution of the problem (1),(2) is oscillatory in G.

Proof. Suppose to the contrary that there exists a nonoscillatory solution $u(x,t) \ne 0$ in $\Omega \times [t_0,+\infty)$ of the problem (1),(2) for some $t_0 \ge 0$. Without loss of generality, we assume that $u(x,t) > 0$, $(x,t) \in \Omega \times [t_0,+\infty)$, $t_0 \ge 0$. By assumption that there exists a $t_1 > t_0$ such that $\tau(t) \ge t_0$, $\sigma(t,\xi) \ge t_0$, $\rho(t,\xi) \ge t_0$ for $(t,\xi) \in [t_1,+\infty) \times [a,b]$, then

$$\left\{\begin{aligned}&u(x,\tau(t)) > 0 \quad for \quad (x,t,\xi) \in \Omega \times [t_1,+\infty),\\ &u(x,\sigma(t,\xi)) > 0 \quad for \quad (x,t,\xi) \in \Omega \times [t_1,+\infty) \times [a,b]\end{aligned}\right.$$

and $u(x,\rho(t,\xi)) > 0$ for $(x,t,\xi) \in \Omega \times [t_1,+\infty) \times [a,b]$.

Proceeding as in the proof of Theorem 3.1, by Lemma 2.1 and from (9), to get that

$$\left.\begin{aligned}&\frac{d}{dt}\left[r(t)\frac{d^{m-1}}{dt^{m-1}}\left(V(t) + c(t)V(\tau(t))\right)\right]\\ &+p(t)\frac{d^{m-1}}{dt^{m-1}}\left(V(t) + c(t)V(\tau(t))\right)\\ &+\int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi)\\ &= -\lambda_0 a(t)V(t) - \lambda_0\int_a^b b(t,\xi)V(\rho(t,\xi))d\eta(\xi)\end{aligned}\right\} \quad \text{............(16)}$$

Set $Z(t)=V(t)+c(t)V(\tau(t))$. Equation (16), can be written as

$$\left(r(t)Z^{(m-1)}(t)\right)'+p(t)Z^{(m-1)}(t)+\int_a^b q(t,\xi)V(\sigma(t,\xi))d\eta(\xi)\le 0,\quad t\ne t_k. \quad …..(17)$$

We have $Z(t)>0$, for $t\ge t_1$ and $\left[r(t)Z^{(m-1)}(t)\right]'<0$ for $t>t_1$. Hence $r(t)Z^{(m-1)}(t)$ is a decreasing in the interval $[t_0,+\infty)$. We can claim that $r(t)Z^{(m-1)}(t)>0$ for $t\ge t_1$. In fact, $r(t)Z^{(m-1)}(t)\le 0$ for $t\ge t_1$, then there exists a $T\ge t_1$ such that $r(T)Z^{(m-1)}(T)<0$. Which implies that

$$\left(r'(t)+p(t)\right)Z^{(m-1)}(t)+r(t)(m-1)Z^{(m-2)}(t)\le 0. \quad ………(18)$$

From (H_1), we have

$$R'(t)=\exp\left(\int_{t_0}^t \frac{r'(s)+p(s)}{r(s)}\right)\int_{t_0}^t \frac{r'(s)+p(s)}{r(s)}ds$$
$$=R(t)\left(\frac{r'(t)+p(t)}{r(t)}\right)$$

and $R(t)>0$, $R'(t)\ge 0$ for $t\ge t_1$. Multiply by $\frac{R(t)}{r(t)}$ on both sides of the equation (18), we have

$$\frac{R(t)}{r(t)}\left(r'(t)+p(t)\right)Z^{(m-1)}(t)+\frac{R(t)}{r(t)}r(t)(m-1)Z^{(m-2)}(t)\le 0$$
$$\left(R(t)Z^{(m-1)}(t)\right)'\le 0. \quad ………(19)$$

From (19), we have

$$R(t)Z^{(m-1)}(t)\le R(T)Z^{(m-1)}(T)\le 0,\quad t\ge T.$$

Thus

$$\int_T^t Z^{(m-1)}(s)ds\le \int_T^t \frac{R(T)}{R(s)}Z^{(m-1)}(T)ds,\quad t\ge T$$
$$\left[Z^{(m-2)}\right]_T^t\le R(T)Z^{(m-1)}(T)\int_T^t \frac{1}{R(s)}ds,\quad t\ge T$$
$$Z^{(m-2)}(t)-Z^{(m-2)}(T)\le R(T)Z^{(m-1)}(T)\int_T^t \frac{1}{R(s)}ds,\quad t\ge T$$
$$Z^{(m-2)}(t)\le Z^{(m-2)}(T)+R(T)Z^{(m-1)}(T)\int_T^t \frac{1}{R(s)}ds,\quad t\ge T.$$

Since (H_1), we see that the right side tends to negative infinity. Thus $\lim_{t\to+\infty} Z^{(m-2)}(t)=-\infty$, which implies $Z(t)$ is eventually negative. This contradicts the face that $Z(t)>0$. At the same time, we can prove $Z^{(m-1)}(t)\ge 0$, $t\ge t_2$. Furthermore, from Lemma 2.2, there exits a $t_2\ge t_1$ and a odd number l, $0\le l\le m-1$, and for $t\ge t_2$, we have

$$Z^{(i)}(t)>0,\qquad 0\le i\le l,$$
$$(-1)^{(i-1)}Z^{(i)}(t)>0,\qquad l\le i\le m-1.$$

By choosing $i=1$, we have $Z'(t)>0$, since $Z(t)\ge x(t)>0$, $Z'(t)\ge 0$, we have $Z(\sigma(t,\xi))\ge Z(\sigma(t,\xi)-\tau(t,\xi))\ge x(\sigma(t,\xi)-\tau(t,\xi))$, and thus

$$\left(r(t)Z^{(m-1)}(t)\right)'+p(t)Z^{(m-1)}(t)+\int_a^b q(t,\xi)Z(\sigma(t,\xi))\left(1-c(\sigma(t,\xi))\right)d\eta(\xi)\le 0.$$

From equation (17), we get

$$\left(r(t)Z^{(m-1)}(t)\right)'+p(t)Z^{(m-1)}(t)+g_0\int_a^b q(t,\xi)Z(\sigma(t,\xi))d\eta(\xi)\le 0. \quad …….(20)$$

From (H_3) and (H_4), we obtain $Z(\sigma(t,\xi))\ge Z(\sigma(t,a))>0$, $\xi\in[a,b]$ *and* $\theta(t)\le\sigma(t,\xi)\le t$.

Thus $Z(\theta(t))\le Z(\sigma(t,a))$ for $t\ge t_2$.

Then (20) can be written as

$$\left(r(t)Z^{(m-1)}(t)\right)'+p(t)Z^{(m-1)}(t)+g_0\int_a^b q(t,\xi)Z(\theta(t))d\eta(\xi)\le 0. \quad …..(21)$$

For $t\ge t_0$, $t=t_k$, $k=1,2,\cdots$ and from (4) we have

$$a_k^{(i)}\le \frac{\frac{\partial^{(i)}Z(x,t_k^+)}{\partial t^{(i)}}}{\frac{\partial^{(i)}Z(x,t_k)}{\partial t^{(i)}}}\le b_k^{(i)}.$$

Therefore $Z(t)$ is an eventually positive solution of (15). This contradicts the hypothesis and completes the proof.

Theorem 3.3. If $\beta(x)\not\equiv 0$ for $x\in\partial\Omega$, there exists a function $\varphi(t)\in C'(\mathbb{R}^+,(0,+\infty))$ for $t_0>0$ which is nondecreasing with respect to t, such that

$$\int_{t_0}^{+\infty}\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[\varphi(s)G(s)-\frac{F^2(s)\varphi(s)}{4L(s)}\right]ds=+\infty, \quad …..(22)$$

then every solution of the boundary value problem (1), (2) is oscillatory in G.

Proof. To prove the solutions of (1), (2) are oscillatory in G, from Theorem 3.2, it is enough to prove that the impulsive differential inequality (15) has no eventually positive solution. Suppose that $Z(t)>0$ is a solution of the inequality (15). Define

$$W(t)=\varphi(t)\frac{r(t)Z^{(m-1)}(t)}{Z(\theta(t))},\quad t\ge t_0, \quad …..(23)$$

then $W(t)\ge 0$ for $t\ge t_0$, and

$$W'(t) \leq \frac{\varphi'(t)}{\varphi(t)} W(t) + \frac{\varphi(t)\left[-p(t)Z^{(m-1)}(t) - g_0 \int_a^b q(t,\xi) Z(\theta(t)) d\eta(\xi)\right]}{Z(\theta(t))} - \frac{\varphi(t)\left(r(t)Z^{(m-1)}(t)\right)Z'(\theta(t))\theta'(t)}{Z(\theta(t))^2}.$$

From $Z^{(m)}(t) \leq 0$, according to Lemma 2.3, we obtain

$$Z'(\theta(t)) \geq M(\theta(t))^{m-2} Z^{(m-1)}(t). \quad .....(24)$$

Thus

$$W'(t) \leq F(t)W(t) - G(t)\varphi(t) - \frac{L(t)}{\varphi(t)} W^2(t)$$

$$W(t_k^+) \leq \frac{b_k^{(m-1)}}{a_k^{(0)}} W(t_k).$$

Define

$$U(t) = \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} W(t).$$

In fact, $W(t)$ is continuous on each interval $(t_k, t_{k+1}]$, and in consideration of $W(t_k^+) \leq \frac{b_k^{(m-1)}}{a_k^{(0)}} W(t_k)$. It follows that for $t \geq t_0$,

$$U(t_k^+) = \prod_{t_0 \leq t_j \leq t_k} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} W(t_k^+) \leq \prod_{t_0 \leq t_j < t_k} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} W(t_k) = U(t_k)$$

and for all $t \geq t_0$,

$$U(t_k^-) = \prod_{t_0 \leq t_j \leq t_{k-1}} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} W(t_k^-) \leq \prod_{t_0 \leq t_j < t_k} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} W(t_k) = U(t_k)$$

which implies that $U(t)$ is continuous on $[t_0, +\infty)$.

$$U'(t) + \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right) \frac{U^2(t)L(t)}{\varphi(t)} + \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(t)\varphi(t) - F(t)U(t)$$

$$= \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} W'(t) + \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right) \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-2} \frac{L(t)}{\varphi(t)} W^2(t)$$

$$+ \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(t)\varphi(t) - \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} F(t)W(t)$$

$$= \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[W'(t) + W^2(t)\frac{L(t)}{\varphi(t)} - W(t)F(t) + G(t)\varphi(t)\right] \leq 0.$$

That is

$$U'(t) \leq -\prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right) \frac{L(t)}{\varphi(t)} U^2(t) + F(t)U(t) - \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(t)\varphi(t). \quad ......(25)$$

Taking

$$X = \sqrt{\prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right) \frac{L(t)}{\varphi(t)}} U(t), \quad Y = \frac{F(t)}{2} \sqrt{\prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \frac{\varphi(t)}{L(t)}},$$

from Lemma 2.4, we have

$$F(t)U(t) - \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right) \frac{L(t)}{\varphi(t)} U^2(t) \leq \frac{F^2(t)\varphi'(t)}{4L(t)} \prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}.$$

Thus

$$U'(t) \leq -\prod_{t_0 \leq t_k < t} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[G(t)\varphi(t) - \frac{F^2(t)\varphi'(t)}{4L(t)}\right]. \quad ....(26)$$

Integrating both sides from t_0 to t, we have

$$U(t) \leq U(t_0) - \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[G(s)\varphi(s) - \frac{F^2(s)\varphi'(s)}{4L(s)}\right] ds.$$

Taking $t \to +\infty$, from (22), we have $\lim_{t \to +\infty} U(t) = -\infty$, which leads to a contradiction with $U(t) \geq 0$.

Theorem 3.4. Assume that $\beta \not\equiv 0$ for $x \in \partial\Omega$, suppose that there exist functions $\varphi(t)$ and $\rho(s) \in C'([0, +\infty], (0, +\infty))$ in which $\varphi(t)$ is nondecreasing. If there exist two functions $H(t,s), h(t,s) \in C'(D, \mathbb{R})$, in which $D = \{(t,s) | t \geq s \geq t_0 > 0\}$, such that

(H_7) $H(t,t) = 0, \quad t \geq t_0; \quad H(t,s) > 0, \quad t > s \geq t_0$,

(H_8) $H_t'(t,s) \geq 0, \quad H_s'(t,s) \leq 0,$

(H_9) $-\frac{\partial}{\partial s}[H(t,s)\rho(s)] - H(t,s)\rho(s)F(s) = h(t,s).$

If

$$\limsup_{t \to +\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \leq t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[G(s)\varphi(s)H(t,s)\rho(s) - \frac{1}{4} \frac{|h(t,s)|^2 \varphi(s)}{L(s)H(t,s)\rho(s)}\right] ds = +\infty, \quad ....(27)$$

then every solution of the boundary value problem (1), (2) is oscillatory in G.

Proof. Assume that the boundary value problem (1), (2) has a nonoscillatory solution $u(x,t)$. Without loss of generality, assume that $u(x,t) > 0,\ (x,t) \in \Omega \times \mathbb{R}^+$. The case for $u(x,t) < 0$ can be considered in the same method. Proceeding as the proof of Theorem 3.3, we have $u(x,\tau(t)) > 0,\ u(x,\sigma(t,\xi)) > 0,$ $u(x,\rho(t,\xi)) > 0$ for $(x,t) \in \Omega \times [t_1,+\infty),\ (x,t,\xi) \in \Omega \times [t_1,+\infty) \times [a,b]$, and

$$U'(t) \le -\prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{L(t)}{\varphi(t)}U^2(t) + F(t)U(t) - \prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(t)\varphi(t)$$

multiplying the above inequality by $H(t,s)\rho(s)$ for $t \ge s \ge T$, and integrating from T to t, we have

$$\int_T^t U'(s)H(t,s)\rho(s)ds \le -\int_T^t \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{L(s)}{\varphi(s)}U^2(s)H(t,s)\rho(s)ds + \int_T^t F(s)U(s)H(t,s)\rho(s)ds - \int_T^t \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(s)\varphi(s)H(t,s)\rho(s)ds. \quad \ldots\ldots(28)$$

Thus, we have

$$\int_T^t \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(s)\varphi(s)H(t,s)\rho(s)ds \le U(T)H(t,T)\rho(T) + \int_T^t |h(t,s)U(s)|ds - \int_T^t \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{L(s)}{\varphi(s)}U^2(s)H(t,s)\rho(s)ds. \quad \ldots\ldots..(29)$$

Put

$$X = \sqrt{\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{L(s)}{\varphi(s)}H(t,s)\rho(s)}\,U(s),$$

$$Y = \frac{1}{2}|h(t,s)|\sqrt{\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\frac{\varphi(s)}{L(s)H(t,s)\rho(s)}},$$

from Lemma 2.4, we attain for $t > T \ge t_0$ that

$$|h(t,s)U(s)| - \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{L(s)}{\varphi(s)}H(t,s)\rho(s)U^2(s) \le \frac{1}{4}\frac{|h(t,s)|^2\varphi(s)}{L(s)H(t,s)\rho(s)}\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}. \quad \ldots\ldots.(30)$$

In addition, from (28) and (30), we have

$$\int_T^t \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(s)\varphi(s)H(t,s)\rho(s)ds - \frac{1}{4}\int_T^t \frac{|h(t,s)|^2\varphi(s)}{L(s)H(t,s)\rho(s)}\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} ds \le U(T)H(t,T)\rho(T) \le H(t,t_0)\rho(T)U(T), \quad t > T \ge t_0. \quad \ldots\ldots(31)$$

Thus

$$\frac{1}{H(t,t_0)}\int_{t_0}^t \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[G(s)\varphi(s)H(t,s)\rho(s) - \frac{1}{4}\frac{|h(t,s)|^2\varphi(s)}{L(s)H(t,s)\rho(s)}\right]ds$$
$$= \frac{1}{H(t,t_0)}\left[\int_{t_0}^T + \int_T^t\right]\left\{\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[G(s)\varphi(s)H(t,s)\rho(s) - \frac{1}{4}\frac{|h(t,s)|^2\varphi(s)}{L(s)H(t,s)\rho(s)}\right]\right\}ds$$
$$\le \frac{1}{H(t,t_0)}\int_{t_0}^T \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(s)\varphi(s)H(t,s)\rho(s)ds + \rho(T)U(T)$$
$$\le \int_{t_0}^T \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(s)\varphi(s)\rho(s)ds + \rho(T)U(T).$$

Letting $t \to +\infty$, we have

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[G(s)\varphi(s)H(t,s)\rho(s) - \frac{1}{4}\frac{|h(t,s)|^2\varphi(s)}{L(s)H(t,s)\rho(s)}\right]ds \le \int_{t_0}^T \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(s)\varphi(s)\rho(s)ds + \rho(T)U(T) < +\infty,$$

which leads to a contradiction with (27).

Remark 3.2. In Theorem 3.4, by choosing $\rho(s) = \varphi(s) \equiv 1$, we have the following corollary.

Corollary 3.1. Assume that the conditions of Theorem 3.4 hold, and

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t \prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[G(s)H(t,s) - \frac{1}{4}\frac{|h(t,s)|^2}{L(s)H(t,s)}\right]ds = +\infty,$$

then every solution of the boundary value problem (1), (2) is oscillatory in G.

Remark 3.3. From Theorem 3.4 and Corollary 3.1, we can attain variety of oscillatory criteria by different choices of the weighted function $H(t,s)$.

For example, choosing $H(t,s)=(t-s)^{n-1}$, $t\ge s\ge t_0$, in which $n>2$ is an integer, then $h(t,s)=(n-1)(t-s)^{(n-3)/2}$, $t\ge s\ge t_0$.

From Corollary 3.1, we have

Corollary 3.2. If there exists an integer $n>2$ such that

$$\limsup_{t\to+\infty}\frac{1}{(t-t_0)^{n-1}}\int_{t_0}^{t}\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[G(s)(t-s)^{n-1}-\frac{1}{4}\frac{(n-1)^2}{(t-s)^2L(s)}\right]ds=+\infty, \qquad \text{………(32)}$$

then every solution of the boundary value problem (1), (2) is oscillatory in G.

Examples

We present couple of examples to point up our results established in Section 3.

Example 4.1. Consider the following equation of the form

$$\begin{cases}\dfrac{\partial}{\partial t}\left(\dfrac{5}{6}\dfrac{\partial^5}{\partial t^5}\left(u(x,t)+\dfrac{1}{5}u\left(x,t-\dfrac{\pi}{2}\right)\right)\right) \\ +\dfrac{10}{3}\dfrac{\partial^5}{\partial t^5}\left(u(x,t)+\dfrac{1}{5}u\left(x,t-\dfrac{\pi}{2}\right)\right)+\dfrac{4}{3}\displaystyle\int_{-\pi/2}^{-\pi/4}u(x,t+2\xi)d\xi \\ =\dfrac{11}{3}\Delta u(x,t)+\dfrac{17}{3}\displaystyle\int_{-\pi/2}^{-\pi/4}\Delta u(x,t+2\xi)d\xi, \quad t\ne t_k,\ k=1,2,\cdots \\ u(x,t_k^+)=\dfrac{k}{k+1}u(x,t_k), \\ \dfrac{\partial^{(i)}}{\partial t^{(i)}}u(x,t_k^+)=\dfrac{\partial^{(i)}}{\partial t^{(i)}}u(x,t_k), \quad i=1,2,3,4,5, \quad k=1,2,\cdots\end{cases} \qquad \text{……..(33)}$$

for $(x,t)\in(0,\pi)\times[0,+\infty)$, with the boundary condition

$$u(0,t)=u(\pi,t)=0, \qquad t\ne t_k. \qquad \text{…….(34)}$$

Here $\Omega=(0,\pi)$, $m=6$, $n=3$, $a_k^{(0)}=b_k^{(0)}=\frac{k}{k+1}$, $a_k^{(i)}=b_k^{(i)}=1$, $i=1,2,3,4,5$, $r(t)=\frac{5}{6}$, $c(t)=\frac{1}{5}$, $p(t)=\frac{10}{3}$, $q(t,\xi)=\frac{4}{3}$, $\tau(t)=t-\frac{\pi}{2}$, $\sigma(t,\xi)=\rho(t,\xi)=t+2\xi$, $a(t)=\frac{11}{3}$, $b(t,\xi)=\frac{17}{3}$, $[a,b]=[-\pi/2,-\pi/4]$, $\eta(\xi)=\xi$, $M=1$, $\theta(t)=t$, $\theta'(t)=1$. Since $t_0=1$, $t_k=2^k$, $g_0=\frac{4}{5}$, $G(s)=\frac{4\pi}{15}$, $L(s)=\frac{6s^4}{5}$. Then hypotheses $(H_1)-(H_7)$ hold, moreover

$$\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0\le t_k<s}\frac{a_k^{(0)}}{b_k^{(i)}}ds=\int_1^{+\infty}\prod_{1<t_k<s}\frac{k}{k+1}ds$$
$$=\int_1^{t_1}\prod_{1<t_k<s}\frac{k}{k+1}ds+\int_{t_1^+}^{t_2}\prod_{1<t_k<s}\frac{k}{k+1}ds+\int_{t_2^+}^{t_3}\prod_{1<t_k<s}\frac{k}{k+1}ds+\cdots$$
$$=1+\frac{1}{2}\times 2+\frac{1}{2}\times\frac{2}{3}\times 2^2+\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times 2^3+\cdots$$
$$=\sum_{n=0}^{+\infty}\frac{2^n}{n+1}=+\infty.$$

Now, the condition (32) reads,

$$\limsup_{t\to+\infty}\frac{1}{(t-1)^2}\int_1^t\prod_{1<t_k<s}\frac{k}{k+1}\left[\frac{4\pi}{15}(t-s)^2-\frac{5}{6s^4(t-s)^2}\right]ds=+\infty.$$

Therefore all the conditions of the Corollary 3.2 are satisfied. Therefore, every solution of equation (33) - (34) is oscillatory in G. In fact $u(x,t)=\sin x\ \sin t$ is such a solution.

Example 4.2. Consider the following equation of the form

$$\begin{cases}\dfrac{\partial}{\partial t}\left(t^4\dfrac{\partial^3}{\partial t^3}\left(u(x,t)+\dfrac{1}{t}u(x,t-\pi)\right)\right) \\ -4t^3\dfrac{\partial^3}{\partial t^3}\left(u(x,t)+\dfrac{1}{t}u(x,t-\pi)\right)+t^2\displaystyle\int_{-\pi}^{0}u(x,t+\xi)d\xi \\ =\left(t^4-t^3+12t-\dfrac{24}{t}\right)\Delta u(x,t)+(3t^2-2)\displaystyle\int_{-\pi}^{0}\Delta u(x,t+\xi)d\xi, \quad t\ne t_k,\ k=1,2,\cdots, \\ u(x,t_k^+)=\dfrac{k}{k+1}u(x,t_k), \\ \dfrac{\partial^{(i)}}{\partial t^{(i)}}u(x,t_k^+)=\dfrac{\partial^{(i)}}{\partial t^{(i)}}u(x,t_k), \quad i=1,2,3, \quad k=1,2,\cdots\end{cases} \qquad \text{…(35)}$$

for $(x,t)\in(0,\pi)\times[0,+\infty)$, with the boundary condition

$$u_x(0,t)+u(0,t)=u_x(\pi,t)+u(\pi,t)=0, \quad t\ne t_k. \qquad \text{…..(36)}$$

Here $\Omega=(0,\pi)$, $m=4$, $n=3$, $a_k^{(0)}=b_k^{(0)}=\frac{k}{k+1}$, $a_k^{(i)}=b_k^{(i)}=1$, $i=1,2,3$, $r(t)=t^4$, $c(t)=\frac{1}{t}$, $p(t)=-4t^3$, $q(t,\xi)=t^2$, $\tau(t)=t-\pi$, $\sigma(t,\xi)=\rho(t,\xi)=t+\xi$, $a(t)=t^4-t^3+12t-\frac{24}{t}$, $b(t,\xi)=3t^2-2$, $[a,b]=[-\pi,0]$, $\eta(\xi)=\xi$, $M=1$, $\theta(t)=t$, $\theta'(t)=1$. Since $t_0=1$, $t_k=2^k$, $g_0=1-\frac{1}{t}$, $G(s)=\pi(t^2-t)$, $L(s)=1$. Then the condition (32) reads,

$$\limsup_{t\to+\infty} \frac{1}{(t-1)^2}\int_1^t \prod_{1<t_k<s} \frac{k}{k+1}\left[\pi(s^2-s)(t-s)^2 - \frac{s^2}{(t-s)^2}\right]ds = +\infty.$$

Therefore all the conditions of the Corollary 3.2 are satisfied. Therefore, every solution of equation (35) - (36) is oscillatory in G. In fact $u(x,t) = e^{-x}\sin t$ is such a solution.

Conclusions

Since several equations of higher order represent almost accurately physical phenomena, it is desirable to study these equations systematically. The obtained oscillation results for equations (1), (2), extends and generalizes some known results in obtained in the area of higher order partial differential equations without impulsive effect and distributed delay. In particular the results are the extensions of the results reported in literature.

Conflict of interest

Authors declare there are no conflicts of interest.

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