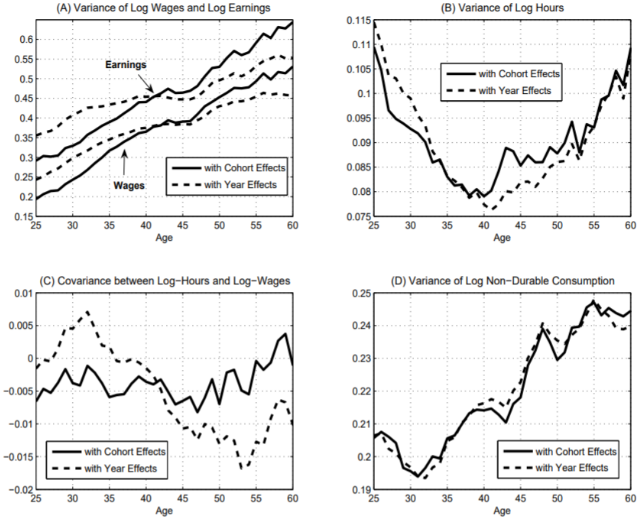
Today

- Introduce the lifecycle dimension into incomplete markets model.
- Key distinction: age becomes a state variable.
  - Adds significant computation complexity.
  - Why do we go through all this trouble?

Review of Empirical Patterns



Source: Heathcote Storesletten Violante (2004)

Review of Empirical Patterns

1. Wage inequality rises linearly over the lifecycle - variance of log wages rises by .3 from 25-55
2. Consumption inequality also rises, but more slowly - variance of nondurable consumption rises by about .07 from 25-55
3. Hours inequality – in contrast – relatively flat or u-shaped
4. Can't do so without introducing lifecycle/age as state

Matching Empirical Patterns

- Patterns robustly documented in a number of studies
- Qualitative patterns hold across countries
- Given such pronounced, seemingly important sources of inequality, natural to want to explore welfare implications (including various counterfactual policies)
- Can't do so without introducing lifecycle/age as state

Outline

- Set-up basic model
- Analyze planner problem (full insurance)
- Analyze autarky (no insurance)
- Analyze incomplete markets model (partial insurance)

Based on Storesletten Telmer Yaron (2004), but framework similar to number of other studies.

Time and Demographics

- Time
  - Time is discrete and indexed by $t = 0, 1, ..., \infty$
- Demographics
  - Economy populated by $J + 1$ overlapping cohorts of individuals at any time t.
  - Each cohort is indexed by time of birth τ and consists of unit measure of individuals
  - Individuals are indexed by i.
  - Born at age $j = 0$ (think age 25) and live to at most age J (think age 60).
  - Survive up to age j with probability ϕ^j.
  - Thus, at any given time t the mass of age j individuals individuals alive is denoted $\phi^{t-\tau}$.

Preferences

- Agents derive utility from consumption and disutility from hours worked.

$$u(c,h) = \frac{c^{1-\gamma}}{1-\gamma} - A\frac{h^{1+\sigma}}{1+\sigma}$$

  - $1/\gamma$ is intertemporal elasticity of substitution
  - $1/\sigma$ is the Frish labor supply elasticity

Endowments

- Individual productivity has a deterministic cohort- and deterministic age component $\kappa_{j,\tau}$
- Individual productivity also has a stochastic, idiosyncratic component $\varepsilon_{i,j,\tau}$
    - $\kappa_{j,\tau}$allows for differences in wage-age profiles and $\varepsilon_{i,j,\tau}$ allows for differences in variance of shocks across cohorts

Technology

- For simplicity, allow for a production technology that is linear in aggregate labor efficiency units:

$$Y = zN$$

  where N is the aggregate efficiency weighted hours worked.

- In other words,

$$N_t = \sum_{\tau=t-J}^{t} \phi^{t-\tau} \int_i n_{i,t-\tau,\tau} d\mu_{t-\tau,\tau}$$

$$n_{i,t-\tau,\tau} = \kappa_{t-\tau,\tau} \epsilon_{i,t-\tau,\tau} h_{i,t-\tau,\tau}$$

Planner's Problem

$$\max_{c_{i,j,\tau},h_{i,j,\tau}} \sum_{\tau=t-J}^{t} \sum_{j=t-\tau}^{J} \beta^{j-(t-\tau)} \phi^{j} \int \lambda_{i,\tau} \left[\frac{c_{i,j,\tau}^{1-\gamma}}{1-\gamma} - A \frac{h_{i,j,\tau}^{1+\sigma}}{1+\sigma} \right] d\mu_{t-\tau,\tau}$$

$s.t.$

$$0 = \sum_{\tau=t-J}^{t} \phi_{t-\tau} \int \left(\kappa_{t-\tau,\tau} \epsilon_{i,t-\tau,\tau} h_{i,t-\tau,\tau} - c_{i,t-\tau,\tau} \right) d\mu_{t-\tau,\tau}$$

- $\lambda_{i,\tau}$ is the weight put on individual i of cohort τ and μ_τ is the cross-sectional distribution of agents in cohort τ.

Planner's Solution - Consumption

- Let θ_t denote the Lagrange multiplier on the resource constraint at date t
- Optimal consumption allocations characterized by:

$$\phi^{t-\tau}\lambda_{i,\tau}c_{i,t-\tau,\tau}^{-\gamma} = \phi^{t-\tau}\theta_t$$

- i.e., the ratio of MU(c) between any two agents i, τ and i', τ' is constant over time and equals the ratio of the planner weights.
    - i.e., Complete markets

Planner's Solution - Consumption

- Taking logs of FOC yields

$$
\begin{aligned}
log\lambda_{i,\tau} - \gamma logc_{i,t-\tau,\tau} &= log\theta_t \\
logc_{i,t-\tau,\tau} &= (log\lambda_{i,\tau} - log\theta_t)/\gamma \\
var_j(logc_{i,t-\tau,\tau}) &= \frac{1}{\gamma^2} var(log\lambda_{i,\tau})
\end{aligned}
$$

- Three takeaways from above equation:

1. Consumption inequality is positive iff planner weights $\lambda_{i,\tau} \neq \lambda_{-i,\tau}$ for some $i \neq -i$
2. Consumption inequality differs across cohorts iff $\lambda_{i,\tau} \neq \lambda_{i,-\tau}$ for some $\tau \neq -\tau$.
3. Consumption inequality is independent of age. Thus, it does not rise over the lifecycle.

Planner's Solution - Labor Supply

- FOC with respect to h:
- Intratemporal FOC for an agent i of cohort τ at date t:

$$\phi^{t-\tau}\lambda_{i,\tau}Ah^{\sigma}_{i,t-\tau,\tau} = \phi^{t-\tau}\theta_t z k_{t-\tau,\tau}\epsilon_{i,t-\tau,\tau}$$

- Again, taking logs yields

$$log h_{i,j,\tau} = \frac{1}{\sigma}\left(log\theta_t + log k_{j,\tau} + log\epsilon_{i,j,\tau} - log\lambda_{i,\tau} + log(z/A)\right)$$

Planner's Solution - Labor Supply

- Taking variance conditional on age yields

$$var_j(log h_{i,j,\tau}) = \frac{1}{\sigma^2}\left(var_j(log \epsilon_{i,j,\tau}) + var(log \lambda_{i,\tau})\right)$$

- Three takeaways from above equation:

1. Hours inequality grows over the lifecycle if $var_j(log \epsilon_{i,j,\tau}$ increases with j.
2. If wage inequality grows steeply however, than hours inequality must grow as well unless σ is very large (Frisch elasticity is very low).
3. However, if σ is big, then the unconditional hours variance is small, in contrast to the data.

Planner's Solution - Summary

Overall, the planner's problem with full insurance does not align with empirical observations.

Autarky

We now turn our attention to analyzing the other extreme case of autarky.

- No insurance, no inter-temporal smoothing, etc.
- Consumption equals earnings each period.

$$c_{i,j,\tau} = \kappa_{j,\tau} \epsilon_{i,j,\tau} h_{i,j,\tau}$$

- Hours FOC:

$$c_{i,j,\tau}^{-\gamma} \kappa_{j,\tau} \epsilon_{i,j,\tau} = A h_{i,j,\tau}^{\sigma}$$

Autarky - Consumption

- Solving for consumption:

$$c_{i,j,\tau} = \left(\frac{1}{A}\right)^{\frac{1}{\gamma+\sigma}} \left(\kappa_{j,\tau}\epsilon_{i,j,\tau}\right)^{\frac{1+\sigma}{\gamma+\sigma}}$$

$$logc_{i,j,\tau} = \frac{1+\sigma}{\gamma+\sigma} log\kappa_{j,\tau} + \frac{1+\sigma}{\gamma+\sigma} log\epsilon_{i,j,\tau} - \frac{1}{\gamma+\sigma} logA$$

- Cross-sectional variance is then:

$$var(logc_{i,j,\tau}) = \left(\frac{1+\sigma}{\gamma+\sigma}\right)^2 var(log\epsilon_{i,j,\tau})$$

Autarky - Consumption

$$var(log c_{i,j,\tau}) = \left(\frac{1+\sigma}{\gamma+\sigma}\right)^2 var(log \epsilon_{i,j,\tau})$$

- Consumption inequality grows over the lifecycle.
  - If $\gamma < 1$, consumption inequality grows more than wage inequality
  - If $\gamma > 1$, consumption inequality grows less than wage inequality

Autarky - Hours

- Solving for hours:

$$c_{i,j,\tau}^{-\gamma}\kappa_{j,\tau}\epsilon_{i,j,\tau} = Ah_{i,j,\tau}^{\sigma}$$

$$(\kappa_{j,\tau}\epsilon_{i,j,\tau}h_{i,j,\tau})^{-\gamma}\kappa_{j,\tau}\epsilon_{i,j,\tau} = Ah_{i,j,\tau}^{\sigma}$$

$$h_{i,j,\tau} = \left(\frac{1}{A}\right)^{\frac{1}{\sigma+\gamma}}(\kappa_{j,\tau}\epsilon_{i,j,\tau})^{\frac{1-\gamma}{\sigma+\gamma}}$$

- Taking logs:

$$logh_{i,j,\tau} = \frac{1-\gamma}{\sigma+\gamma}log\kappa_{j,\tau} + \frac{1-\gamma}{\sigma+\gamma}log\epsilon_{i,j,\tau} - \frac{1}{\sigma+\gamma}logA$$

- Cross-sectional variance is then:

$$var(logh_{i,j,\tau}) = \left(\frac{1-\gamma}{\sigma+\gamma}\right)^2 var(log\epsilon_{i,j,\tau})$$

Autarky - Hours

$$var(logh_{i,j,\tau}) = \left(\frac{1-\gamma}{\sigma+\gamma}\right)^2 var(log\epsilon_{i,j,\tau})$$

- Again, wage inequality increases over the lifecycle, so inequality in log hours will increase unless $\gamma = 1$.
  - If $\gamma << 1$, then hours inequality is likely to grow rapidly
  - If $\gamma >> 1$, then hours inequality will grow very little
- Takeaway: if $\gamma >> 1$ then qualitative patterns of lifecycle (e.g., consumption inequality grows less than wage inequality, hours inequality is relatively flat) can be matched reasonably well.
  - Is this reasonable?

Plausible Lifecycle Economy

- Storesletten Telmer Yaron (2004)
- Similar to above, but some differences:
  - Only look at stationary economy so no τ.
  - No labor supply decision

Time and Demographics

- Time
  - Time is discrete and indexed by $t = 0, 1, ..., \infty$
- Demographics
  - Economy populated by $J + 1$ overlapping cohorts of individuals at any time t.
  - Each period a unit measure of individuals are born
  - Individuals are indexed by i.
  - Born at age $j = 0$ (think age 25) and live to at most age J (think age 90).
  - Survive up to age j with probability ϕ^j – conditional survival $\phi_j = \phi^j / \phi^{j-1}$.

Preferences

- Agents derive utility from consumption.

$$u(c, h) = \frac{c^{1-\gamma}}{1-\gamma}$$

  - $1/\gamma$ is intertemporal elasticity of substitution

Endowments

- Individual productivity has a deterministic age component κ_j
- Individual productivity also also has a stochastic, idiosyncratic component $\varepsilon_{i,j}$ which follows an AR(1)

$$
\begin{aligned}
\log y_{i,j} &= \kappa_j + \varepsilon_{i,j} \\
\varepsilon_{i,j} &= \rho \varepsilon_{i,j-1} + \omega_{i,j}
\end{aligned}
$$

- μ_j denotes age j distribution of productivity.
  - $\pi(\varepsilon_j, \varepsilon_{j-1})$ denotes the discretized Markov transition
- Retirement:
  - $\varepsilon_j = 0$ if $j \geq J^{ret}$

Technology

- Standard production function, CES with capital and labor inputs

$$Y = F(K, N)$$

- Since labor supply is inelastic, total labor is

$$N = \sum_{j=0}^{J^{Ret}-1} \phi^j \int y_{i,j} d\mu_j$$

- Aggregate capital is

$$K = \sum_{j=0}^{J} \phi^j \int a_{i,j+1} d\mu_j$$

- Aggregate resource constraint

$$C + \delta K = F(K, N)$$

Government

- Taxes labor income at rate τ. So receipts are denoted as

$$\tau wN$$

- Finances a pay-as-you-go social security system.
  - Income in retirement is a function of average lifetime productivity (stationary assumption so no need to track earnings).

$$\bar{y}_i = \sum_{j=0}^{J^{Ret}-1} y_{i,j}$$

  - Income is redistributed progressively according to function $P(\bar{y})$ where $P' > 0$, $P'' < 0$.
- Estates of individuals that die early are fully taxed and redistributed across all agents via lump-sum transfer φ

Assets

- One period non-state-contingent bonds are traded
- Agents can borrow up to borrowing limit $\bar{a}$. Must be able to repay in all future contingencies.
- Asset and good markets are competitive.

Household problem

Non-retired

$$
\begin{aligned}
V_j(\varepsilon_j, a_j, \bar{y}_j) &= \max_{c_j, a_{j+1}} u(c_j) + \beta\psi_{j+1}\mathbb{E}\left[V_{j+1}(\varepsilon_j, a_j, \bar{y}_{j+1})\right] \\
&\quad s.t. \\
c_j + a_{j+1} &= Ra_j + (1-\tau)wy_j + \varphi \\
a_{j+1} &\geq -\bar{a} \\
\bar{y}_{j+1} &= \bar{y}_j + y_j / J^{Ret}
\end{aligned}
$$

Retired

$$
\begin{aligned}
V_j(0, a_j, \bar{y}_j) &= \max_{c_j, a_{j+1}} u(c_j) + \beta\psi_{j+1} V_{j+1}(0, a_j, \bar{y}_{j+1}) \\
&\quad s.t. \\
c_j + a_{j+1} &= Ra_j + P(\bar{y}) + \varphi \\
a_{j+1} &\geq -\bar{a}
\end{aligned}
$$

Equilibrium

A stationary equilibrium is:

- A set of state-contingent decision rules $\{c_j(s), a_{j+1}(s)\}_{j=0}^{J}$
- A set of value functions $\{V_j(s)\}_{j=0}^{J}$
- Prices w and R
- Aggregate capital and labor K and N
- Taxes τ and transfers φ
- And stationary measures $\{\mu_j\}_{j=0}^{J}$

Equilibrium

Such that

- Decision rules solve value functions
- Prices equal marginal products of capital and labor
- The labor market clears
- The capital market clears
- Government budgets balanced

$$\tau w N = \sum_{j=J^{Ret}}^{J} \psi^j \int P(\bar{y}_{i,j}) d\mu_j$$

$$\sum_{j=0}^{J} (1-\psi_j) \int a_{i,j} d\mu_j = \varphi \sum_{j=0}^{J} \psi^j$$

Equilibrium

Such that

- Goods market clears
- Distributions are stationary across time (not necessarily age!)

Solution

Similar to what we did before.

- Big difference: Now need to solve for $J+1$ decision rules and value functions
- To do so, we use backward induction:
  - Given prices, taxes, and transfers, we can solve age $J-1$ decision problem.
  - Given age $J-1$ value function and decision rules, can solve age $J-2$ decision problem
  - ...
  - Given age 1 value function and decision rules, can solve age 0 decision problem
- EGM is very well suited for this type of problem!

General Solution Method

1. Guess R and φ.
2. Given R, we can obtain the aggregate capital K.
3. Given K and N (exogenous) we can obtain w.
4. Given w recover τ.
5. Now, we can solve for $J+1$ decision rules and value functions.
6. Simulate a large number of households.
   - Start at age 0, simulate through age J using decision rules from previous step.
   - Calculate (unconditional) average capital and unintentional bequests for each age. Add over ages to obtain measures of $\hat{K}$ and $\hat{\varphi}$.
7. Adjust interest rate (increase if $\hat{K} < K$, decrease if $\hat{K} > K$) and go back to step 1.

Results

Given assumed earning/hours inequality, model matches consumption risk quite well.

Results

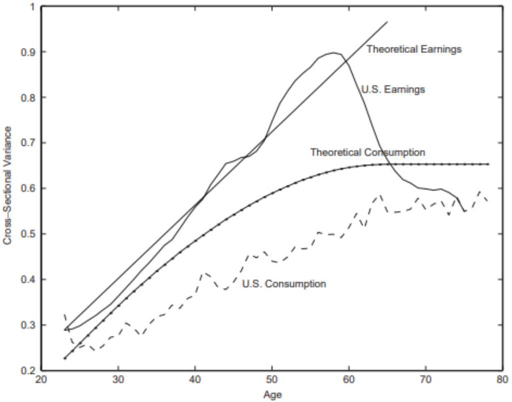


Fig. 4. This graph compares the population moments from our benchmark model with those of the data. The solid lines (without dots) represent the theoretical and empirical cross-sectional variance of log earnings. The dashed line represents the empirical cross-sectional variance of consumption and the solid-dotted line represents the theoretical cross-sectional variance of consumption from the benchmark economy.

Results

- Choice of β
  - In baseline, $\beta = .962$ matches $K/Y = 3.5$ ratio.
  - Reducing $\beta = .93$ so that $K/Y = 1.5$ significantly increases consumption inequality
  - More capital helps agents smooth out shocks, reducing consumption inequality.
- Social security
  - Social security redistributes wealth, provides consumption insurance.
  - Necessary to match consumption inequality exactly.

Results

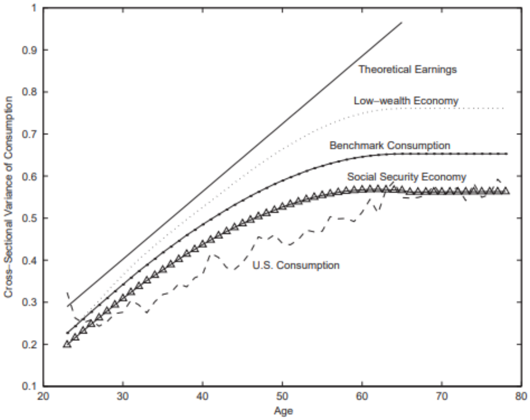


Fig. 5. The dashed line represents the empirical cross-sectional variance of consumption. The solid-dotted line represents the theoretical variance for the benchmark economy, whereas the dotted and triangle-marked lines represent the low wealth and social security economies, respectively, discussed in Section 4 of the text. All the economies have an identical pattern of earnings inequality, given by the solid line (which closely matches its empirical counterpart).

Aggregate Risk

- In above example we got rid of all aggregate risk. Krueger Kubler (2003) show how to include!
- What are the key differences?
  - We now have to solve for $J + 1$ decision rules value functions, all of which depend on z

$$Y = zF(K, N).$$

  - Stationary distribution doesn't exist. So aggregate state is a function of $J + 1$ distributions μ_j
- Obviously this is impossible.

Aggregate Risk

- Krusell-Smith relied on linearity of consumption function.
- However, we know (and can check) that consumption won't be linear over the lifecycle.
  - If you only track K in this economy obtain an $R^2 = .66$.
- Krueger Kubler solution: Track wealth distribution as a parametric function of age!
  - Assume no idiosyncratic risk, i.e., Aggregate state characterized by $\bar{a} = [a_0, ..., a_J]$
  - Track wealth distribution as a parametric function of age!
  - Find that in most cases a cubic polynomial results in good aggregation for most calibrations.