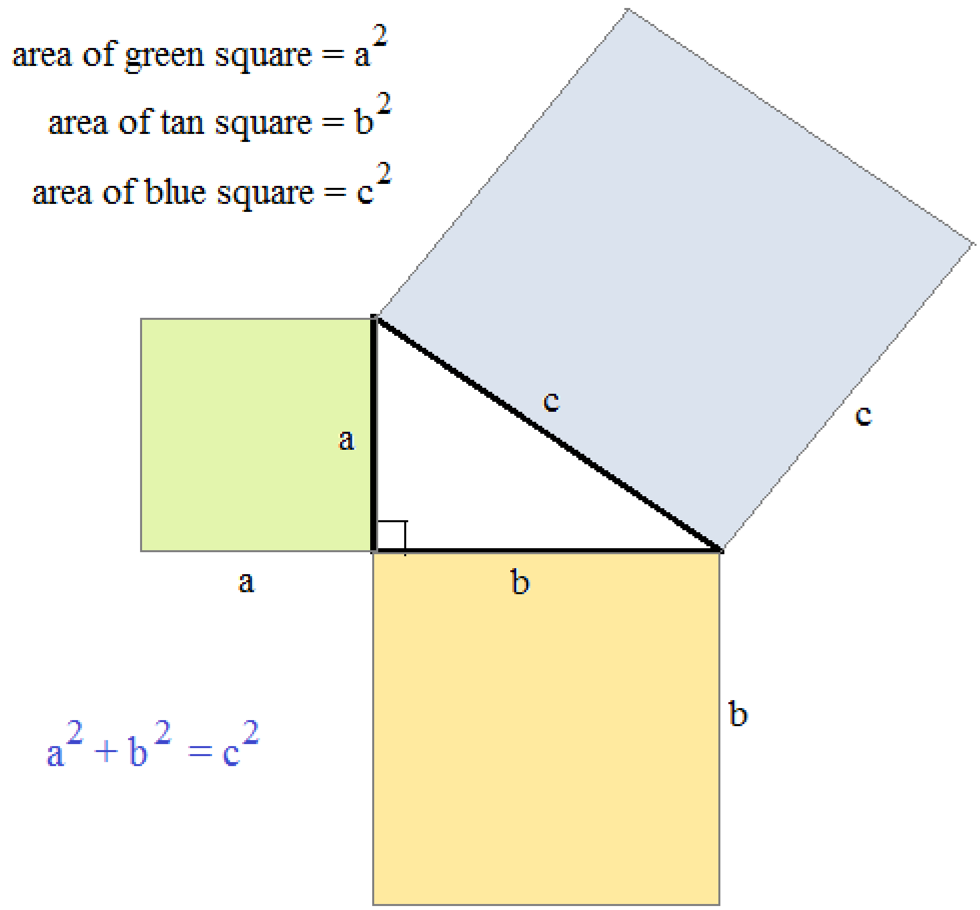
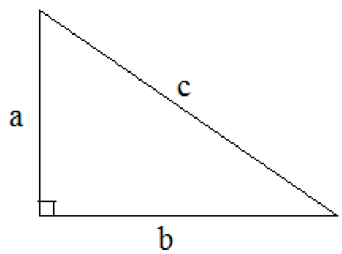
Pythagorean Theorem & Distance

Notes, proofs, examples, and test (w/solutions)



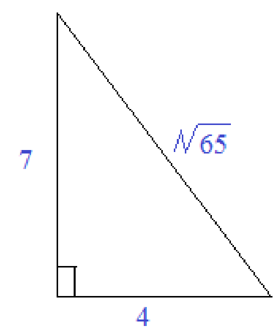
Pythagorean Theorem:

$a^2 + b^2 = c^2$ where a and b are lengths of the legs of a right triangle and c is the length of the hypotenuse



"sum of the squares of the legs is equal to the square of the hypotenuse"

Example:



$(4)^2 + (7)^2 = c^2$

$16 + 49 = 65$

$c = \sqrt{65}$

Identifying triangles by their sides:

$a^2 + b^2 = c^2$	right triangle
$a^2 + b^2 > c^2$	acute triangle
$a^2 + b^2 < c^2$	obtuse triangle

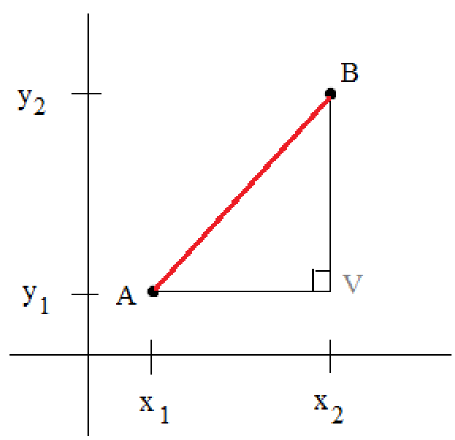
Distance Formula illustrates Pythagorean Theorem!

$AB^2 = AV^2 + BV^2$

point A: (x_1, y_1)

point B: (x_2, y_2)

distance AB $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$



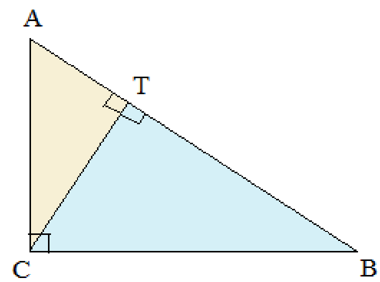
There are many ways to prove (verify) the Pythagorean Theorem.
Here are 2 approaches:

1) Using Proportional Triangles:

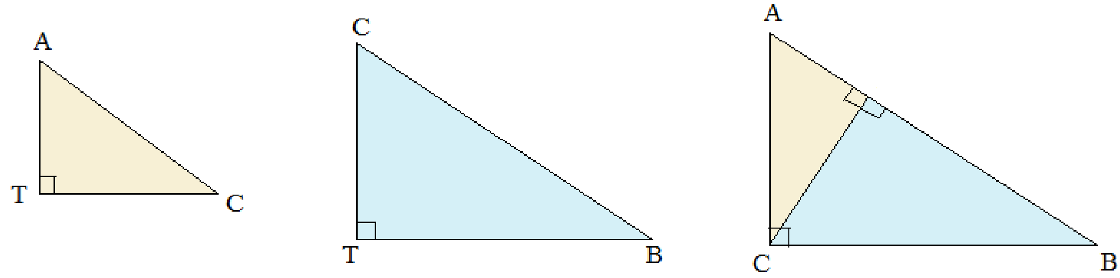
$\triangle$ ABC is a right triangle

$\overline{CT}$ is an altitude

(an altitude drawn from the vertex of a right triangle to the hypotenuse forms three similar right triangles)



Let's divide into 3 triangles and compare:



1)

$$\frac{AC}{AB} = \frac{AT}{AC} \qquad \frac{CB}{TB} = \frac{AB}{CB}$$

$$\frac{\text{Hypotenuse 1}}{\text{Hypotenuse 3}} = \frac{\text{Left leg 1}}{\text{Left leg 3}} \qquad \frac{\text{Bottom leg 3}}{\text{Bottom leg 2}} = \frac{\text{Hypotenuse 3}}{\text{Hypotenuse 2}}$$

2) (Cross multiply each proportion)

$(AC)(AC) = (AB)(AT)$

$(CB)(CB) = (AB)(TB)$

3) (Add them together and simplify)

$(AC)(AC) + (CB)(CB) = (AB)(AT) + (AB)(TB)$

$(AC)^2 + (CB)^2 = (AB)[(AT) + (TB)]$

$(AT) + (TB) = (AB)$

$(AC)^2 + (CB)^2 = (AB)^2$

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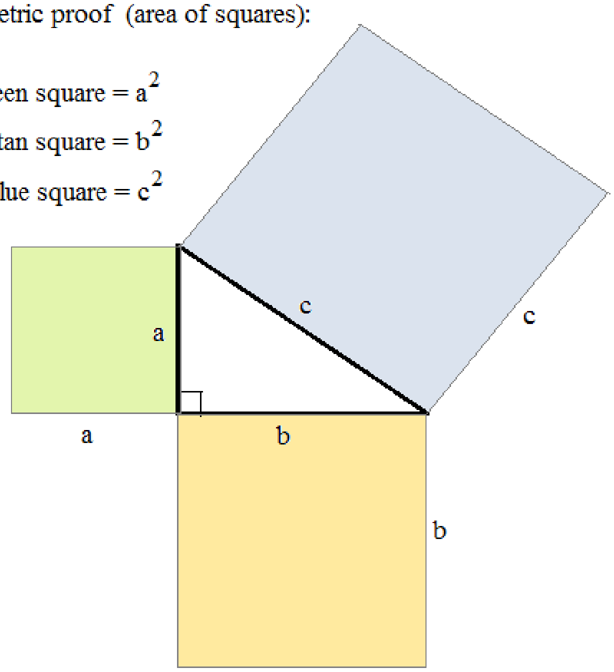
2) Geometric proof (area of squares):

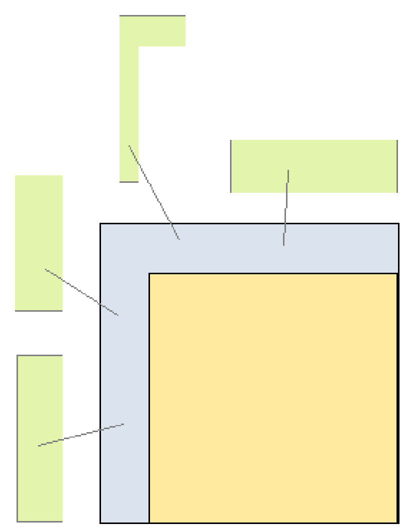
area of green square = a^2

area of tan square = b^2

area of blue square = c^2



**area of blue square will equal area of tan square plus area of green square

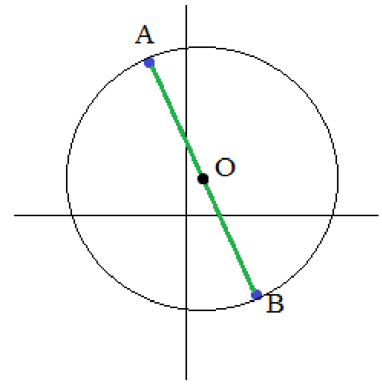


Distance Formula and Pythagorean theorem

Example: A and B are endpoints of a diameter of circle O.

A: (−1, 5)
B: (3, −3)

What is the area of the circle?



Step 1: Draw a diagram and identify formulas

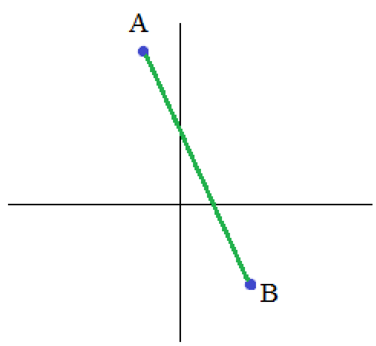
$$\text{Area} = \pi(\text{radius})^2$$

$$\text{radius} = \frac{1}{2}(\text{diameter})$$

Step 2: Find missing variable(s)

We need to find the distance from A to B

$$\text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



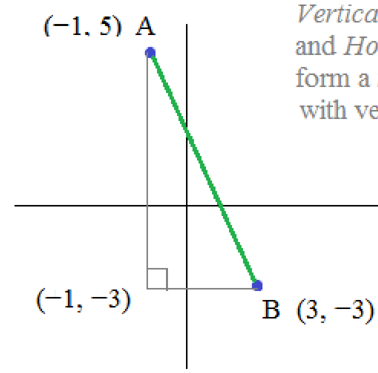
$$d\overline{AB} = \sqrt{(-1-3)^2 + (5-(-3))^2}$$

$$= \sqrt{16+64} = \sqrt{80} = \boxed{4\sqrt{5}}$$

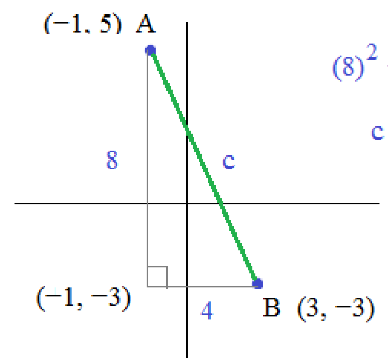
OR

We need to find the length of $\overline{AB}$

Pythagorean Theorem: $a^2 + b^2 = c^2$



Vertical line from A and *Horizontal* line from B form a *right triangle* with vertex (x_A, y_B)



$$(8)^2 + (4)^2 = c^2$$

$$c = \sqrt{80} = \boxed{4\sqrt{5}}$$

Step 3: Answer question

$$\text{Area} = \pi(\text{radius})^2$$

Since the diameter AB = $4\sqrt{5}$,
the radius of the circle is $2\sqrt{5}$

Then, the area of the circle is $\pi(2\sqrt{5})^2 = 20\pi$

Note: The distance formula and Pythagorean Theorem are quite similar!

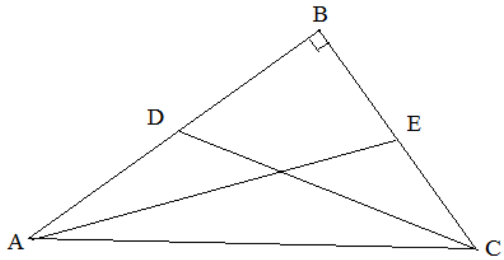
Given: Right Triangle ABC

$\overline{AE}$ and $\overline{CD}$ are medians

$\overline{AE} = 4\sqrt{10}$

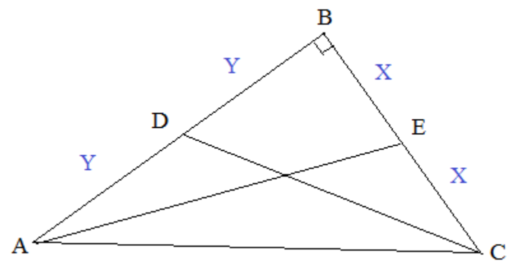
$\overline{CD} = 10$

Find the length of $\overline{AC}$



Since $\overline{AE}$ is a median, $\overline{BE} = \overline{CE}$

$\overline{CD}$ is a median, $\overline{AD} = \overline{BD}$



Since $\angle B$ is a right angle,

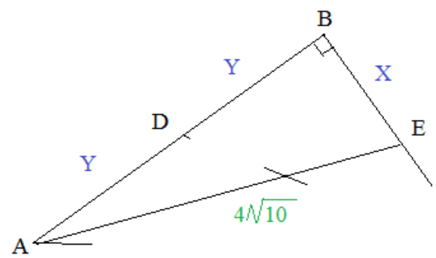
$\triangle$ CBD is a right triangle

$\triangle$ABE is a right triangle

Use pythagorean theorem to find X and Y:

$$X^2 + (2Y)^2 = (4\sqrt{10})^2$$

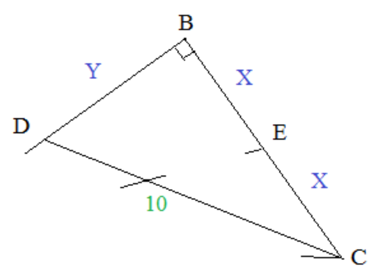
$$X^2 + 4Y^2 = 160$$



$$(2X)^2 + Y^2 = 10^2$$

$$4X^2 + Y^2 = 100$$

2 equations with 2 unknowns: use substitution to find solutions...



$$4(160 - 4Y^2) + Y^2 = 100$$

$$640 - 16Y^2 + Y^2 = 100$$

$$-15Y^2 = -540$$

$$Y^2 = 36$$

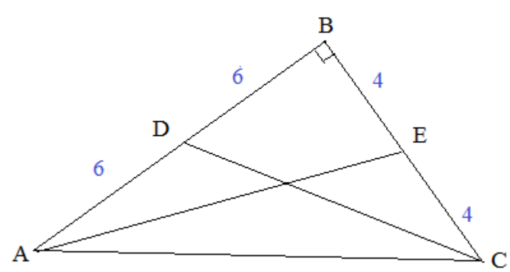
$Y = 6$, ~~-6~~ (since we're measuring length, we'll eliminate the negative value)

$$4X^2 + Y^2 = 100$$

$$4X^2 + 6^2 = 100$$

$$4X^2 = 64$$

$X = 4$, ~~-4~~



AB = 12

BC = 8

$$\overline{AC} = \sqrt{AB^2 + BC^2}$$

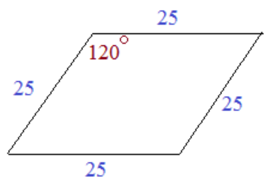
$$= \sqrt{144 + 64} = \boxed{4\sqrt{13}}$$

Pythagorean Theorem, Right Angle, and Distance Examples

Example: The perimeter of a rhombus is 100 inches.
One of the interior angles is 120 degrees.

What are the lengths of the diagonals?

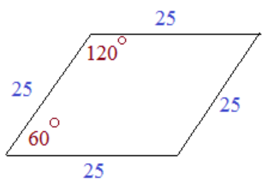
Step 1: Draw a picture and label



Rhombus: all sides are congruent

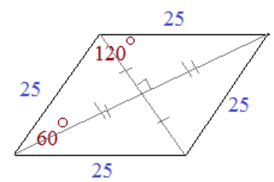
each side = $\frac{100 \text{ inches}}{4 \text{ sides}}$ = 25 inches

Step 2: Develop equation

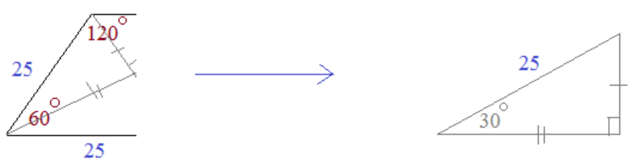


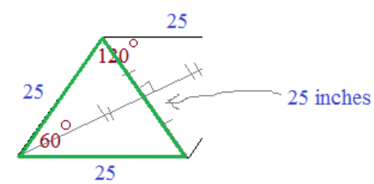
Rhombus: diagonals are *perpendicular bisectors*

since opposite sides are parallel, then the adjacent sides are *supplementary*



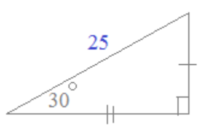
Each of the 4 triangles inside the rhombus is a 30-60-90 right triangle!





Note: one of the diagonals is the 3rd side of an equilateral triangle

Step 3: Solve and Answer Question



Since the hypotenuse is 25 inches,

'small side' is 25/2 = 12.5 inches

'medium side' is 12.5 x $\sqrt{3} \cong$ 21.65 inches

Therefore, the long diagonal is 2 x 21.65 = (approx)

43.3 inches

the short diagonal is 2 x 12.5 =

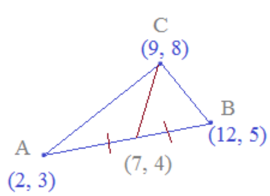
25 inches

Example: The vertices of triangle ABC are the coordinates

A = (2, 3)
B = (12, 5)
C = (9, 8)

What is the length of the *median* from point C to side $\overline{AB}$?

Step 1: Sketch a diagram



Step 2: Find relevant equation

Definition of a median: segment drawn from vertex to midpoint of the opposite side....

What's the midpoint of $\overline{AB}$?

$$\left(\frac{2+12}{2}, \frac{3+5}{2} \right) = (7, 4)$$

Step 3: Solve and Answer question

The length of the median is the distance from C to the median of AB.

$$\text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$d_{\text{median}} = \sqrt{(9-7)^2 + (8-4)^2}$$

$$= \sqrt{4 + 16} = \boxed{2\sqrt{5}}$$

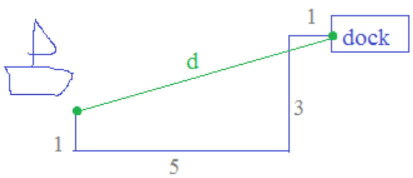
Pythagorean Theorem, Right Angle, and Distance Examples

Example: A boat leaves the dock and goes 1 mile west; then, 3 miles south; then, 5 miles west; then 1 mile north...

How far is the boat from the dock?

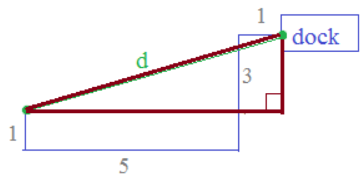
Step 1: Draw a picture and label

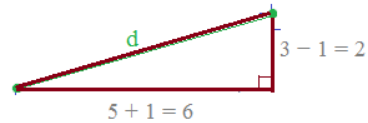
find d:



Step 2: Set up the equation

The trick to setting up the equation is recognizing that the boat has ultimately formed a right triangle!





Step 3: Solve

Since we know the legs of the right triangle, we can use the Pythagorean Theorem to find the hypotenuse (d):

$$(6)^2 + (2)^2 = d^2$$

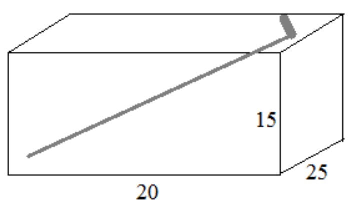
$$40 = d^2$$

The boat is $\sqrt{40}$

or, approx. 6.3 miles from the dock.

Example: You have a cardboard box with dimensions 20" x 15" x 25".
If you want to ship a 33" golf club, can you fit the club inside your cardboard box and mail it?

Step 1: Draw a diagram and label

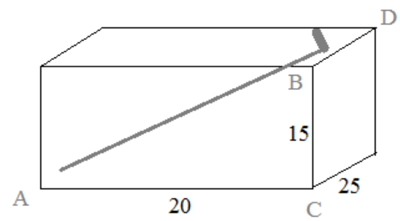


Step 2: Develop strategy and equations

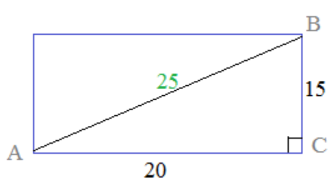
Obviously, the 33" club must be placed diagonally inside the box to fit (because 33 > 20, 15, or 25)...

So, what is the maximum length inside the box?
i.e. what is the diagonal from one corner to the opposite corner of the box?

Since every vertex/corner of the box is a right angle, we can use the pythagorean theorem to find lengths.



Step 3A: Find the 'front' diagonal

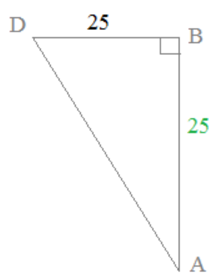


Pythagorean theorem (or recognize that 15-20-25 is a multiple of 3-4-5)

$$15^2 + 20^2 = 25^2 \quad = \overline{AB}$$

Step 3B: Find the main diagonal across the box

(looking above the box)



Using Pythagorean theorem
(or recognize that this is a 45-45-90 triangle)

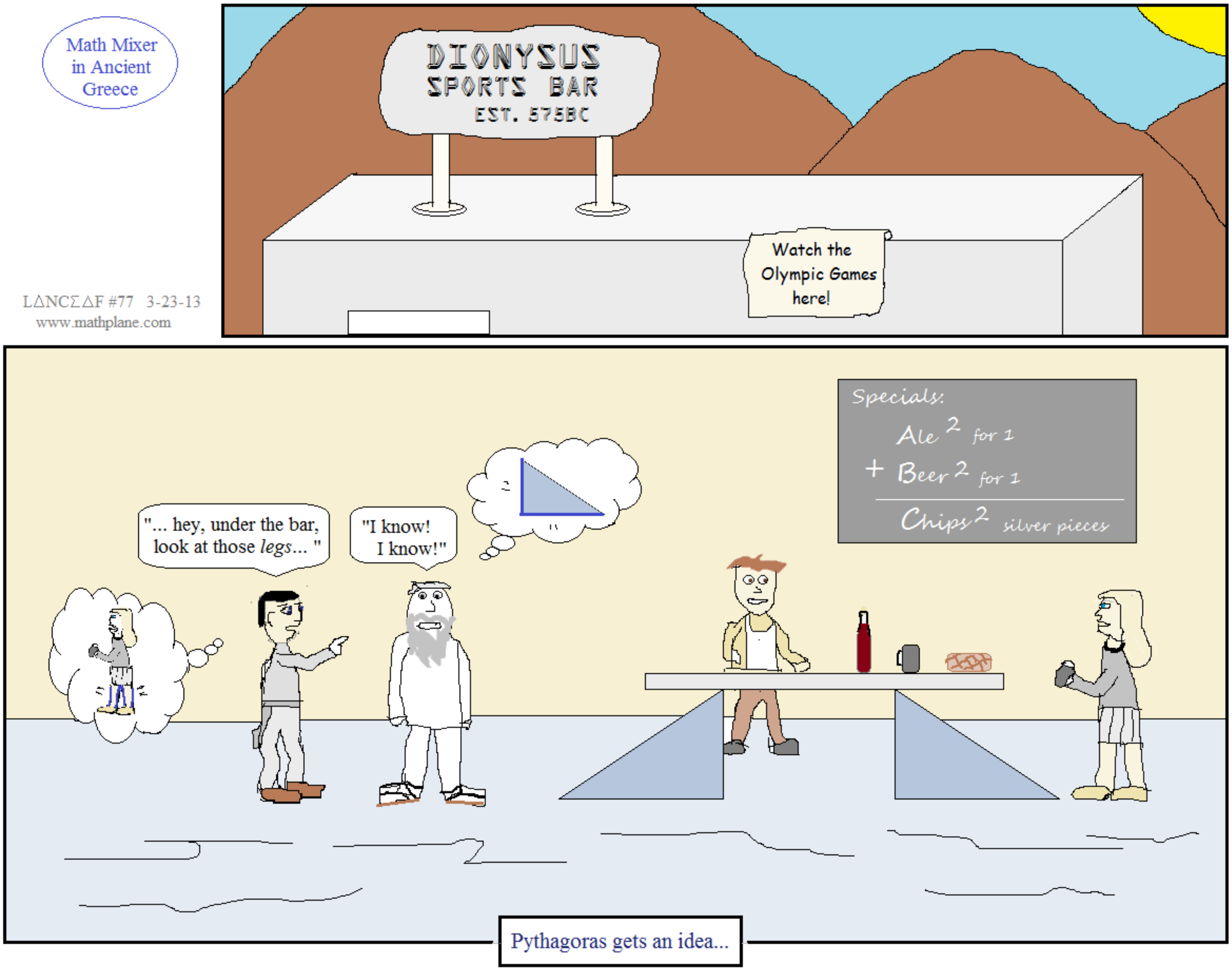
$$\overline{AD} = 25\sqrt{2}$$

Step 4: Solve/Answer the question

The length of the diagonal $\overline{AD}$ is

$25\sqrt{2}$ inches or approximately 35.35"

Since this distance exceeds the length of the golf club, the club will fit inside the box!



Practice Quiz on next page...

Right triangle, pythagorean theorem, and distance questions

Part I: Formulas and Definitions

1) Given the lengths of the sides of a triangle, determine if each is *right, acute, obtuse, or neither*:

   a) 30, 40, 50

   b) 3, 7, 12

   c) 6, 8, 11

   d) 2, 2, 2

   e) 4, 6, 8

2) Find the lengths of segments with endpoints:

   a) (-1, 6) and (3, 4)

   b) (2, -4) and (2, 9)

   c) (-3, -5) and (0, 0)

Part II: Applying Geometry Concepts

1) Find the altitude of a trapezoid with sides 2, 41, 20, and 41 respectively..

Right triangle, pythagorean theorem, and distance questions

2) Given: Triangle ABC

Coordinates:

A = (2, 3)
B = (3, 7)
C = (6, 1)

a) Find the length of the *median* from B to $\overline{AC}$:

b) Find the length of the *altitude* from A to $\overline{BC}$:

3) If the endpoints of a hypotenuse are (-2, 3) and (5, -4), identify *two possible* vertices of the right triangle.

Geometry Quiz: Pythagorean Theorem, Right Triangles, & Distance

Part III: More Geometry Applications

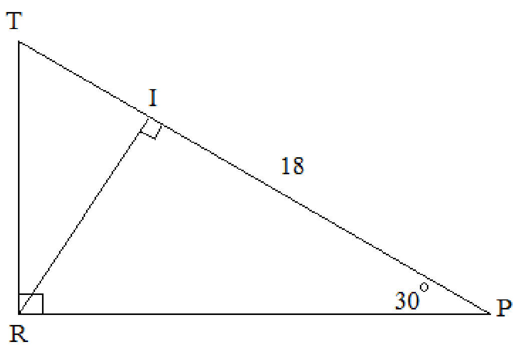
1) TRP is a right triangle

$\overline{PI} = 18$

$\angle P = 30^\circ$

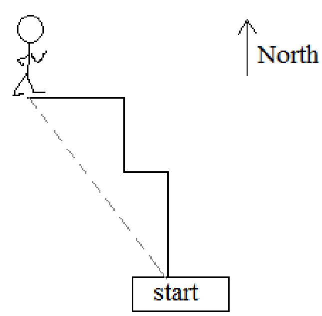
$\overline{RI}$ is an altitude

Find the perimeter of $\triangle TRI$



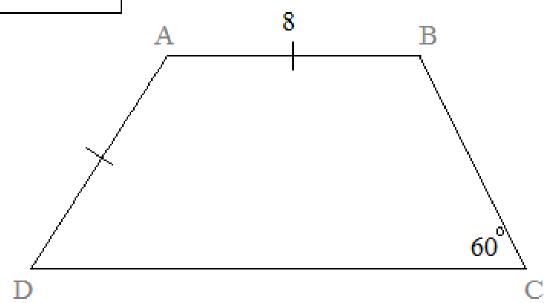
2) You have a box where the length, width, and depth are no longer than 2'6".
If you want to ship a golf club that is 4'5", would the club fit inside the box?

3) A racer runs 5 miles north, 2 miles west, 3 miles north, and 4 miles west.
How far is he from the starting line?



Geometry Quiz: Pythagorean Theorem, Right Triangles, & Distance

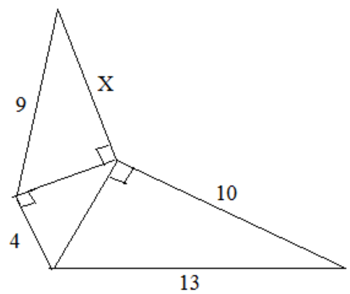
4) ABCD is a (*non*-isosceles) trapezoid. (see diagram)
If the length of the altitude is 6, find $\overline{CD}$.



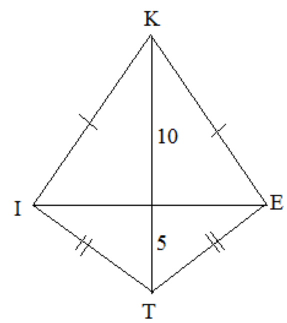
5) What is the *area* of an equilateral triangle with perimeter 30 meters?

6) The point (5, n) is *equidistant* from (1, 3) and (10, 2).
Find n.

7) Find X



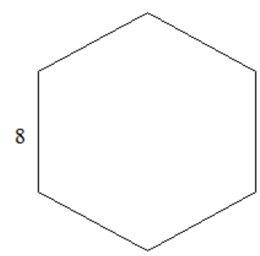
8) If KIT and KET are right angles, what is the perimeter of KITE?



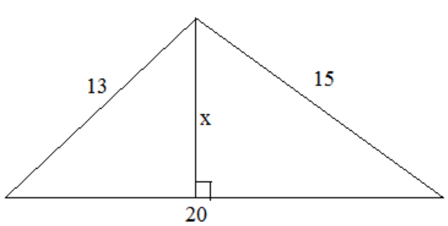
9) If the figure is a regular hexagon,

a) how many diagonals?

b) what is the sum of the lengths of all the diagonals?



10) What is the length of altitude x ?



Right triangle, pythagorean theorem, and distance questions

SOLUTIONS

Part I: Formulas and Definitions

1) Given the lengths of the sides of a triangle, determine if each is *right, acute, obtuse, or neither*:

a) 30, 40, 50 Right (10 x 3-4-5 triangle)

b) 3, 7, 12 Neither (does not exist because $3 + 7 < 12$)

c) 6, 8, 11 Obtuse $36 + 64 < 121$

d) 2, 2, 2 Acute (also, equilateral)

e) 4, 6, 8 Obtuse $16 + 36 < 64$

$a^2 + b^2 = c^2$ then right triangle

$a^2 + b^2 > c^2$ then acute triangle

$a^2 + b^2 < c^2$ then obtuse triangle

2) Find the lengths of segments with endpoints:

a) (-1, 6) and (3, 4)

$$\sqrt{(6-4)^2 + (-1-3)^2} = \sqrt{4+16} = \boxed{2\sqrt{5}}$$

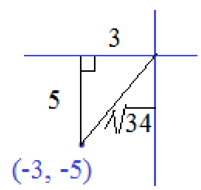
$$\text{distance} = \sqrt{(y_1 - y_2)^2 + (x_1 - x_2)^2}$$

b) (2, -4) and (2, 9) (vertical line segment)

$\boxed{13 \text{ units}}$

c) (-3, -5) and (0, 0)

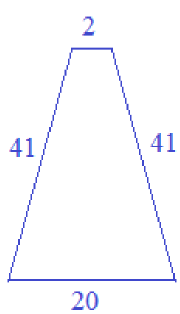
$$\sqrt{(0+3)^2 + (0+5)^2} = \sqrt{9+25} = \boxed{\sqrt{34}}$$

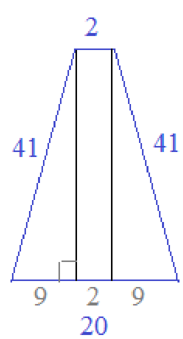


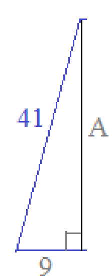
Part II: Applying Geometry Concepts

1) Find the altitude of a trapezoid with sides 2, 41, 20, and 41 respectively..

Since it is a trapezoid, 2 sides must be parallel...







Use Pythagorean Theorem to find the altitude A:

$$(9)^2 + (A)^2 = (41)^2$$

$$81 + (A)^2 = 1681$$

$$\boxed{A = 40}$$

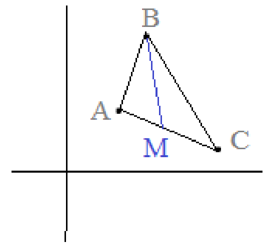
Right triangle, pythagorean theorem, and distance questions

SOLUTIONS

2) Given: Triangle ABC

Coordinates:

A = (2, 3)
B = (3, 7)
C = (6, 1)



a) Find the length of the *median* from B to $\overline{AC}$:

Step 1: Draw a sketch
Step 2: Identify the median (from B to the *midpoint* of $\overline{AC}$)
Step 3: Find coordinates

B = (3, 7)   midpoint M = $\left(\frac{2+6}{2}, \frac{3+1}{2}\right) = (4, 2)$

Step 4: Find distance between coordinates

Use distance formula

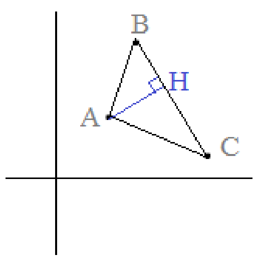
length of median

$\overline{BM} = \sqrt{(4-3) + (2-7)} = \boxed{\sqrt{26}}$

b) Find the length of the *altitude* from A to $\overline{BC}$:

Note: The altitude is perpendicular to the base.

To find point H, we need to find the intersection of $\overline{AH}$ and $\overline{BC}$



1) Line BC: slope between (3, 7) and (6, 1)

$\frac{7-1}{3-6} = -2$

then, line segment

$y - 1 = -2(x - 6)$

$\boxed{y = -2x + 13}$

2) Line AH: slope is 1/2
(opposite reciprocal of BC slope)

then, line segment

$y - 3 = 1/2(x - 2)$

$\boxed{y = 1/2(x) + 2}$

3) $y = -2x + 13$
$y = 1/2(x) + 2$

$-2x + 13 = 1/2(x) + 2$

$x = 22/5$
then, $y = 21/5$

4) Finally, find distance from A to H:

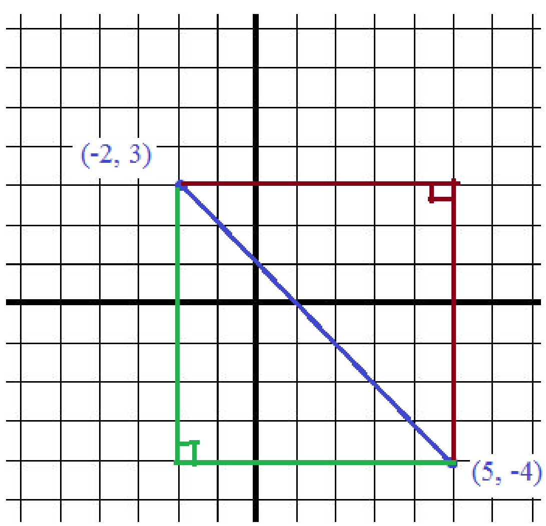
(22/5, 21/5)
(2, 3)

$\sqrt{(12/5)^2 + (6/5)^2}$

$\sqrt{180/25} = \boxed{\frac{6\sqrt{5}}{5}}$

(approx. 2.68)

3) If the endpoints of a hypotenuse are (-2, 3) and (5, -4), identify *two possible* vertices of the right triangle.



(5, 3)

or

(-2, -4)

Geometry Quiz: Pythagorean Theorem, Right Triangles, & Distance

SOLUTIONS

Part III: More Geometry Applications

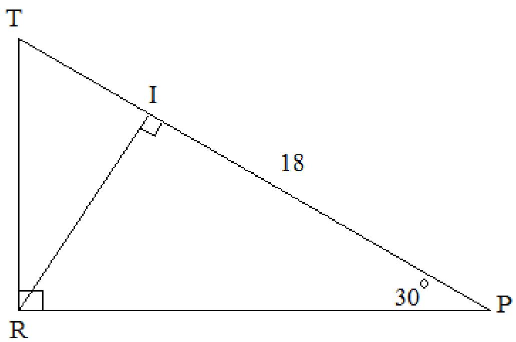
1) TRP is a right triangle

$\overline{PI} = 18$

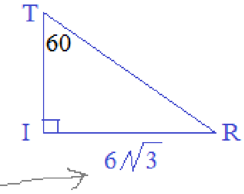
$\angle P = 30^\circ$

$\overline{RI}$ is an altitude

Find the perimeter of $\triangle$TRI



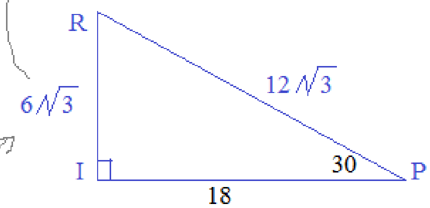
TRP is a right triangle; RI is an altitude from the vertex to the hypotenuse... Therefore, there are 3 similar right triangles!



Since angle P is 30 degrees, we know the other angles are 60 degrees.

(We have three 30-60-90 triangles)

$x - \sqrt{3}x - 2x$



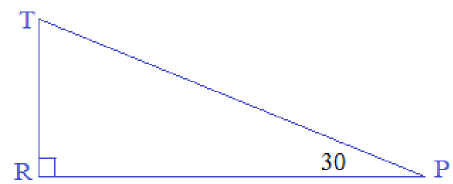
If PI = 18, then

$RI = \frac{18}{\sqrt{3}} = 6\sqrt{3}$

Since $RI = 6\sqrt{3}$

TI = 6

RT = 12

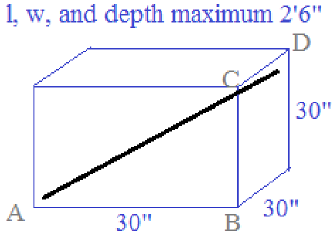


Perimeter $\triangle TRI = 18 + 6\sqrt{3}$

2) You have a box where the length, width, and depth are no longer than 2'6". If you want to ship a golf club that is 4'5", would the club fit inside the box?

Step 1: Draw a diagram; identify variables and formulas

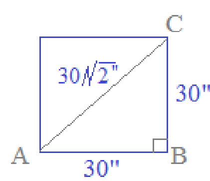
Assume l, w, and depth maximum 2'6"



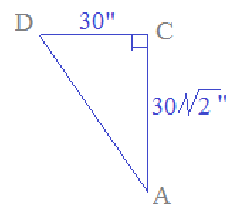
The maximum length is the diagonal of the box

Step 2A: Find front diagonal

Use Pythagorean Theorem or 45-45-90 ratios



Step 2B: Find the cross diagonal (front left to back right)

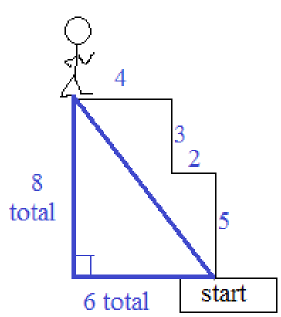


AD = $\sqrt{900 + 1800}$

= 51.96"

Since the golf club (53") exceeds the maximum length inside the box, the club <u>will not</u> fit!!

3) A racer runs 5 miles north, 2 miles west, 3 miles north, and 4 miles west. How far is he from the starting line?



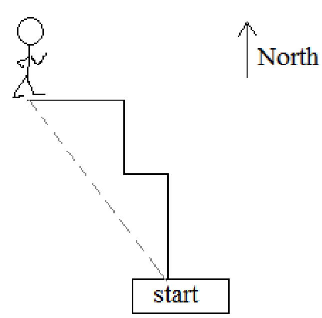
right triangle:

6 - 8 - x

x = 10

("pythagorean triplet")

The racer is 10 miles from the starting line.



Geometry Quiz: Pythagorean Theorem, Right Triangles, & Distance

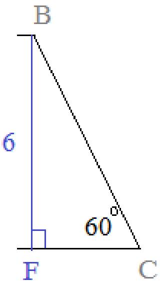
4) ABCD is a (*non*-isosceles) trapezoid. (see diagram)

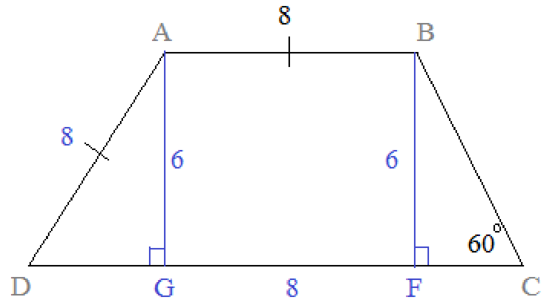
If the length of the altitude is 6, find $\overline{CD}$.

Step 1: Find $\overline{FC}$ 30-60-90 right triangle:

$FC = \dfrac{6}{\sqrt{3}}$

$BC = \dfrac{12}{\sqrt{3}}$



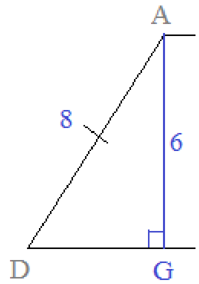


Step 2: Find $\overline{DG}$

use pythagorean thm.

$(8)^2 = (DG)^2 + (6)^2$

$DG = \sqrt{28}$



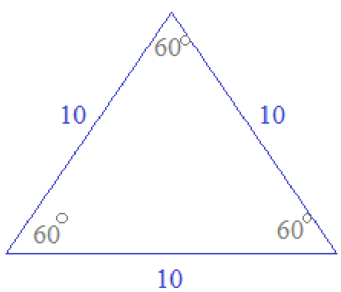
Step 3: Add all 3 parts of the base

$\overline{DC} = \overline{DG} + \overline{GF} + \overline{FC}$

$2\sqrt{7} + 8 + 2\sqrt{3}$

5) What is the *area* of an equilateral triangle with perimeter 30 meters?

Step 1: Draw a picture and label



Equilateral triangle has 3 equal sides and equal angles!

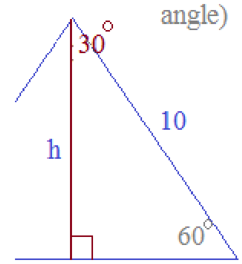
Step 2: Identify formula and find missing variable(s)

Area of $\triangle = \frac{1}{2}$ (base)(height)

base = 10 meters

height = ?

(altitude forms a right angle)



30-60-90 triangle

small side = 1/2 hypotenuse

= 1/2 (10) = 5 meters

medium side = $\sqrt{3}$ small side

$= 5\sqrt{3}$

Step 3: Solve

$\frac{1}{2}$ (base)(height) =

$\frac{1}{2}(10m)(5\sqrt{3}\ m) =$

$25\sqrt{3}$ square meters

6) The point (5, n) is *equidistant* from (1, 3) and (10, 2).
Find n.

$\text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

distance from (5, n) to (1, 3):

$\sqrt{(5-1)^2 + (n-3)^2} =$

$\sqrt{16 + n^2 - 6n + 9}$

distance from (5, n) to (10, 2):

$\sqrt{(5-10)^2 + (n-2)^2} =$

$\sqrt{25 + n^2 - 4n + 4}$

must be equal

Square both equations and combine like terms:

$n^2 - 6n + 25 = n^2 - 4n + 29$

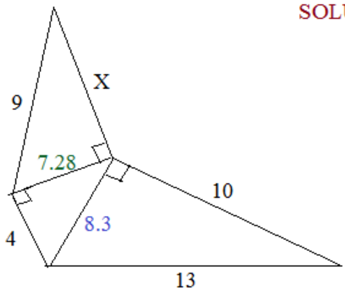
$-2n = 4$

$n = -2$

Geometry Quiz: Pythagorean Theorem, Right Triangles, & Distance

SOLUTIONS

7) Find X



$10^2 + B^2 = 13^2$

$B = \sqrt{69} = 8.3$

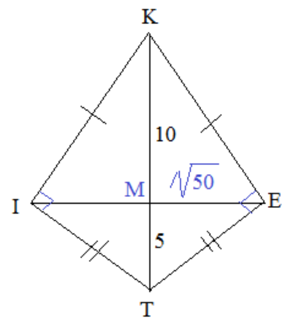
$4^2 + b^2 = 8.3^2$

$b = \sqrt{53} = 7.28$

$7.28^2 + X^2 = 9^2$

$X = \sqrt{28}$

8) If KIT and KET are right angles, what is the perimeter of KITE?



Since $\overline{EM}$ is an altitude (to hypotenuse),

$\overline{EM} = \sqrt{50}$

(Then, using Pythagorean Theorem)

Then, $\overline{ET}$ and $\overline{IT}$ are $5\sqrt{3}$

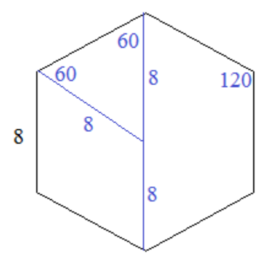
And, $\overline{KE}$ and $\overline{IK}$ are $5\sqrt{6}$

Total: $10\sqrt{3} + 10\sqrt{6}$

9) If the figure is a regular hexagon,

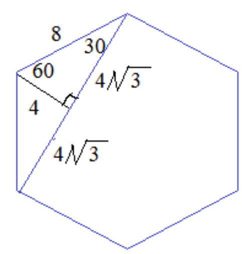
a) how many diagonals?

b) what is the sum of the lengths of all the diagonals?



long diagonals are 16

$\text{diagonals} = \frac{n(n-3)}{2} = \frac{6(3)}{2} = $ 9 diagonals



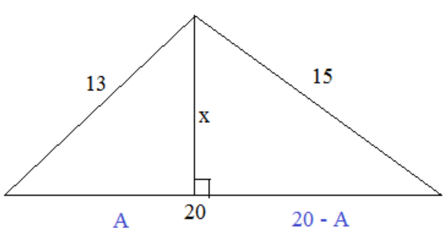
short diagonals are $8\sqrt{3}$

3 of them are 'long diagonals' across,
and 6 of them are 'small diagonals that connect every other vertex...

$3 \times 16 = 48$ $\quad$ $6 \times 8\sqrt{3} = 48\sqrt{3}$

$48 + 48\sqrt{3}$

10) What is the length of altitude x ?



$A^2 + x^2 = 13^2$ Pythagorean Theorem

$(20 - A)^2 + x^2 = 15^2$

$400 - 40A + A^2 + x^2 = 15^2$ Solve the System

$A^2 + x^2 = 13^2$

$400 - 40A = 56$

$40A = 344$

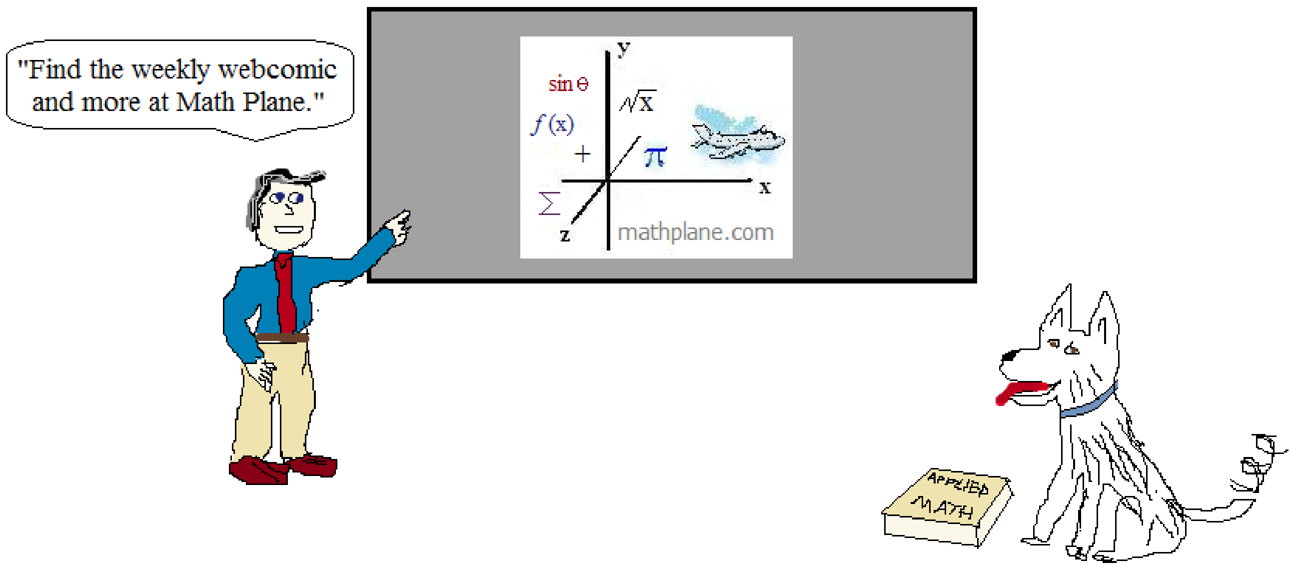
$A = 8.6$

therefore, $x = 9.75$ (approx)

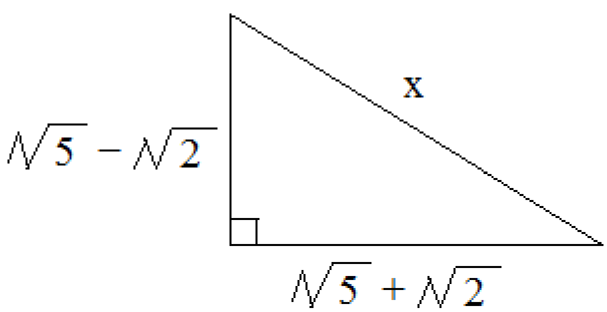
Thanks for visiting the site. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers…



One more question:



What is x?   (Solution on next page)

What is the length x ?

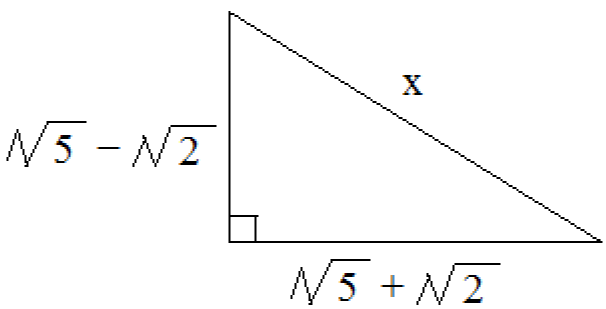
a) $\sqrt{7}$

b) $\sqrt{10}$

c) $4\sqrt{10}$

d) $\sqrt{14}$

e) 7



SOLUTION

Use Pythagorean Theorem to find x

$$a^2 + b^2 = c^2$$

$$(\sqrt{5} + \sqrt{2})^2 + (\sqrt{5} - \sqrt{2})^2 = x^2$$

$$(\sqrt{5} + \sqrt{2})(\sqrt{5} + \sqrt{2}) + (\sqrt{5} - \sqrt{2})(\sqrt{5} - \sqrt{2}) = x^2$$

$$5 + \sqrt{10} + \sqrt{10} + 2 \qquad 5 - \sqrt{10} - \sqrt{10} + 2 = x^2$$

$$5 + \cancel{\sqrt{10}} + \cancel{\sqrt{10}} + 2 \qquad 5 - \cancel{\sqrt{10}} - \cancel{\sqrt{10}} + 2 = x^2$$

$$14 = x^2$$

$$\boxed{\sqrt{14} = x}$$