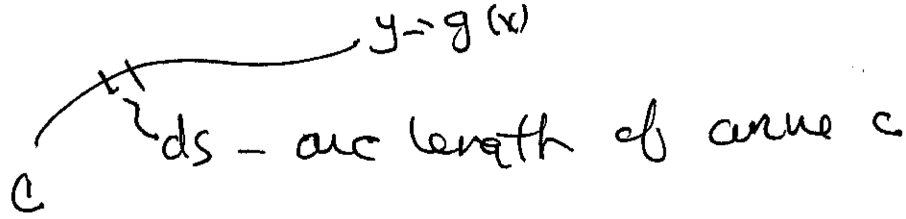
Math 2471 – Calc 3

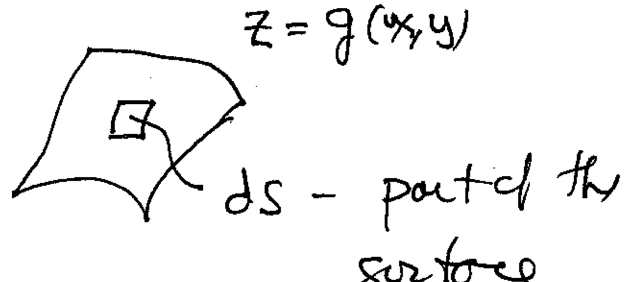
Line integrals

$$\int_C f(x,y)\,ds$$



Surface integrals

$$\iint_S f(x,y,z)\,dS$$



As $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

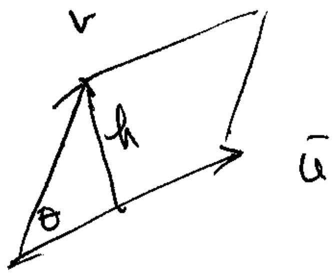
we need something for dS that brings in the surface $z = g(x,y)$

Some things from Calc 2

(i) $\bar{u} \cdot \bar{v} = \|\bar{u}\|\,\|\bar{v}\| \cos\theta$

(ii) $\|\bar{u} \times \bar{v}\| = \|\bar{u}\|\,\|\bar{v}\| \sin\theta$ ← we need this

Now if

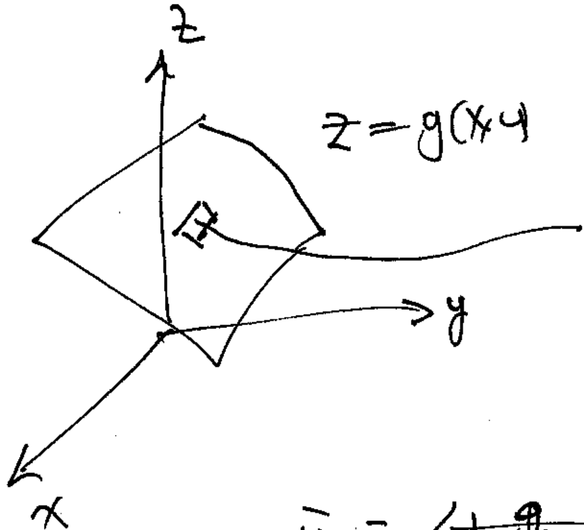


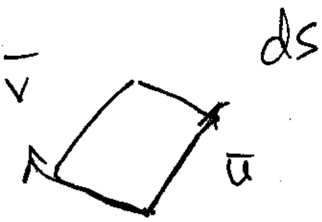
area of this parallelogram is $\|\bar u\| h$

Now $\sin\theta = \dfrac{h}{\|\bar v\|}$ so $A = \|\bar u\|\|\bar v\|\sin\theta$.

So $A = \|\bar u \times \bar v\|$

Now let's go to the surface





create 2 vectors

$\bar u = \cancel{\langle 1, g_x, 0\rangle}$ $\quad \bar v = \langle 0, 1, g_y\rangle dy$

$= \langle 1, 0, g_x\rangle dx$

this is like we did for the tangent plane except for the dx, dy

so $\bar{u} \times \bar{v} = \begin{vmatrix} i & j & k \\ dx & 0 & g_x dx \\ 0 & dy & g_y dy \end{vmatrix}$

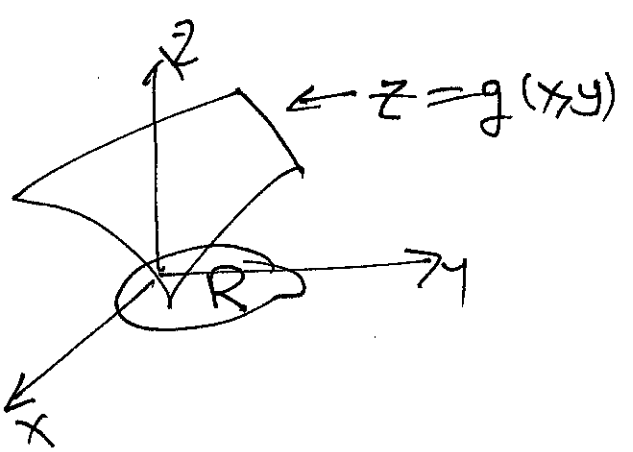
$= \langle -g_x\,dx\,dy, -g_y\,dx\,dy, dx\,dy \rangle$

$\|\bar{u} \times \bar{v}\| = \sqrt{1+g_x^2+g_y^2}\,dx\,dy$

this is $dS = \sqrt{1+g_x^2+g_y^2}\,dx\,dy$

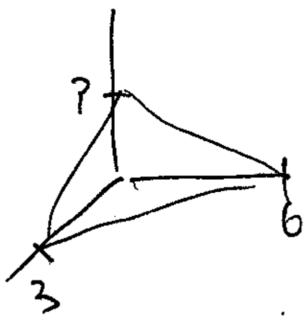
so surface $\int$

$$\iint_R f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}\,dA$$



Evaluate $\iint_S (y^2+2yz)\,dS$

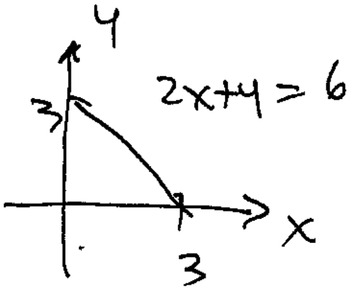
where S is surface of plane $2x+y+2z=6$ 1^{st} oct



$$z = \frac{6-2x-y}{2} = g$$

$$g_x = -1 \quad g_y = -\frac{1}{2}$$

so $$dS = \sqrt{1+1+\frac{1}{4}}\,dA = \sqrt{\frac{9}{4}}\,dA = \frac{3}{2}\,dA$$



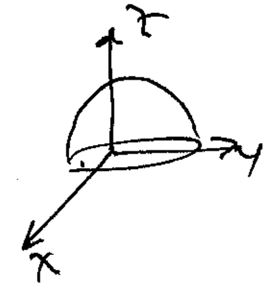
$$\frac{3}{2}\int_0^3\int_0^{6-2x} y^2 + \cancel{2}y\frac{(6-2x-y)}{\cancel{2}}\,dy\,dx$$

$$6y-2xy$$

$$= \frac{3}{2}\int_0^3 3y^2 - xy^2\Big|_0^{6-2x} dx = \frac{3}{2}\int_0^3 3(6-2x)^2 - x(6-2x)^2\,dx$$

$$= \frac{3}{2}\int_0^3 (108-108x+36x^2-4x^3)\,dx = \frac{3}{2}\cdot 81 = \frac{243}{2}$$

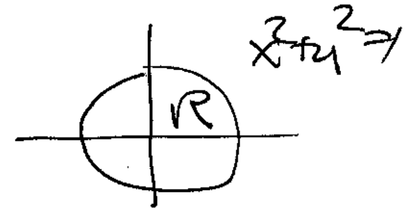
Ex 2 $\iint_S y\,dS$ when S is $z = 1-x^2-y^2$, $z \geq 0$



so $g = 1-x^2-y^2$

$g_x = -2x \quad g_y = -2y$

$$\iint_R y^2\sqrt{1+4x^2+4y^2}\,dA$$



switch to polar

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(r\sin\theta)^2\sqrt{1+4r^2}\,r\,dr\,d\theta$$

$$\int_0^{2\pi}\sin^2\theta\left(\int_0^1 r^3\sqrt{1+4r^2}\,dr\right)d\theta = \underbrace{\int_0^{2\pi}\sin^2\theta\,d\theta}_{=\pi}\cdot\left(\frac{5\sqrt{5}}{24}+\frac{1}{120}\right)$$

$$\left(\frac{5\sqrt{5}}{24}+\frac{1}{120}\right)\pi$$