Math 3331 - ODE's

First order ODEs

$$\frac{dy}{dx} = F(x,y)$$

(1) Separable $\quad \frac{dy}{dx} = f(x)g(y)$

separate $\quad \frac{dy}{g(y)} = f(x)dx$

and $\int$ $\quad \int \frac{dy}{g(y)} = \int f(x)dx + c$

2) Linear $\quad a(x)\frac{dy}{dx} + b(x)y = g(x)$

S.F. $\quad \frac{dy}{dx} + p(x)y = q(x)$

integrating factor $\quad \mu = e^{\int p(x)dx}$

$$\frac{d}{dx}(\mu p) = \mu q$$ ← separate and $\int$

Ex $\frac{dy}{dx} + \tan x \, y = \cos^2 x$, $y(0)$

SF yes $p(x) = \frac{\sin x}{\cos x}$ $\mu = e^{\int \frac{\sin x}{\cos x} dx} = e^{-\ln|\cos x|}$

$= \frac{1}{\cos x}$

$\frac{1}{\cos x} \frac{dy}{dx} + \frac{\sin x}{\cos^2 x} y = \frac{\cos^2 x}{\cos x}$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{\cos x} y\right) = \cos x$

$\int d\left(\frac{1}{\cos x} y\right) = \int \cos x \, dx + c$

$\frac{1}{\cos x} y = \sin x + c$

$y(0) = 2$ $\frac{2}{\cos(0)} = \sin(0) + c \Rightarrow c = 2$

Sol^n $\boxed{y = \sin x \cos x + 2\cos x}$

3. Bernoulli

$$\frac{dy}{dx} + P(x)y = q(x)y^n, \quad n \in \mathbb{R}$$

note: $n = 0$ ↗ linear
$n = 1$ ↗ sep.

Ex $\frac{dy}{dx} + \frac{y}{x} = y^2$ $\frac{dy}{dx} + \frac{y}{x} = e^y$

$$\underbrace{\frac{dy}{dx} + \frac{y}{x} = \frac{1}{y}}$$ this is not

these are Bernoulli

So what ~~m~~ makes these ~~Both~~ Bernoulli here

the y^n, so divide by it

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\frac{y}{y^2} = 1 \qquad y\frac{dy}{dx} + \frac{1}{x}\cdot y \cdot y = 1$$

simplify

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = 1 \qquad y\frac{dy}{dx} + \frac{1}{x}y^2 = 1$$

let $u = \frac{1}{y}$ 2^{nd} ch F $u = y^2$

Ex 1 $\frac{dy}{dx} + y/x = y^2$

(1) divide by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x} \cdot \frac{y}{y^2} = 1$$

$$\underline{\underline{\frac{1}{y^2}\frac{dy}{dx}}} + \frac{1}{x} \cdot \frac{1}{y} = 1$$

let $u = \frac{1}{y} = y^{-1}$ $\frac{du}{dx} = -1y^{-2}\frac{dy}{dx} = -\underline{\underline{\frac{1}{y^2}\frac{dy}{dx}}}$

the terms = look the same so

$$\frac{1}{y^2}\frac{dy}{dx} = -\frac{du}{dx}$$

so $-\frac{du}{dx} + \frac{1}{x} u = 1$ ← linear

$\frac{du}{dx} - \frac{u}{x} = -1$ SF linear

$$\mu = e^{-\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{1}{x} u\right) = -\frac{1}{x} \quad \text{sep.}$$

$$\frac{1}{x} u = -\ln|x| + c$$

Now $u = \frac{1}{y}$

so $\frac{1}{x} \cdot \frac{1}{y} = -\ln|x| + c$

solve for y so $y = \dfrac{1}{x\,(c - \ln|x|)}$

Ex 2 $\quad \dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{y}$

$$y\frac{dy}{dx} + \frac{1}{x} y^2 = 1$$

let $u = y^2 \quad \dfrac{du}{dx} = 2y\dfrac{dy}{dx}$

$$\frac{1}{2}\frac{du}{dx} + \frac{1}{x} u = 1 \quad \text{linear}$$

If linear

$$\frac{du}{dx} + \frac{2}{x}u = 2$$

$$\mu = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$\frac{d}{dx}(x^2 u) = 2x^2$$

$$x^2 u = \frac{2}{3}x^3 + c$$

$u = y^2$ so

$$x^2 y^2 = \frac{2}{3}x^3 + c \qquad \text{sol}^n$$

I would leave this alone.

use IC/BC if you have one.

$y(1) = 0 \Rightarrow 0 = \frac{2}{3} + c \Rightarrow c = -2/3$

$$x^2 y^2 = \frac{2}{3}(x^3 - 1)$$