Math 3331 - ODE's

So far we have considered solving

$$ay'' + by' + cy = 0, \quad a, b, c \text{ const.}$$

We seek sol^n of the form

$$y = e^{mx}$$

where $am^2 + bm + c = 0$

Cases i) $m = m_1, m_2$ real distinct

sol^n $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

ii) $m = m_1, m_1$ repeated real

$$y = c_1 x e^{m_1 x} + c_2 e^{m_1 x}$$

iii) $m = \alpha \pm \beta i$ complex

$$y = c_1 e^{\alpha x} \cos\beta x + c_2 e^{\alpha x} \sin\beta x$$

c_1, c_2 found with I.C's.

Now we solve

$$ay'' + by' + cy = f(x) \quad \text{non homo.}$$

Sol^n has 2 parts

$$y = y_c + y_p$$

y_c - sol^n of $ay'' + by' + cy = 0$

y_p - a particular sol^n of entire ODE.

Methods

(1) method of undetermined coefficients

- good guessing

Ex $y'' - 2y' + y = 2e^x + 8e^{-x}$

y_c: $m^2 - 2m + 1 = 0$

$m = 1, 1$. $y_c = c_1 x e^x + c_2 e^x$

for $2e^x$ part bump twice e^{-x}

$$y_p = Ax^2 e^x + Be^{-x}$$

Now substitute so

$$y_p' = 2Axe^x + Ax^2e^x - Be^{-x}$$

$$y_p'' = 2Ae^x + 4Axe^x + Ax^2e^x + Be^{-x}$$

and

$$2Ae^x + \cancel{4Axe^x} + \cancel{Ax^2e^x} + Be^{-x}$$
$$- \cancel{4Axe^x} - \cancel{2Ax^2e^x} + 2Be^{-x}$$
$$+ \cancel{Ax^2e^x} + Be^{-x} = 2e^x + 8e^{-x}$$

$\Rightarrow$ $2A = 2$, $4B = 8$

$A = 1$ $B = 2$

$$y_p = x^2e^x + 2e^{-x}$$

as $y = c_1xe^x + c_2e^x + x^2e^x + 2e^{-x}$

2nd method – Reduction of Order

Given 1 sol^n of $ay'' + by' + cy = 0$, say y_1

let $y = y_1 u$ (a method we've seen before)

Same ex., $y_1 = e^x$

$$y = e^x u$$

$$y' = e^x u' + e^x u, \quad y'' = e^x u'' + 2e^x u' + e^x u$$

sub $y'' - 2y' + y = 2e^x + 8e^{-x}$

$$e^x u'' + \cancel{2e^x u'} + \cancel{e^x u}$$
$$- \cancel{2e^x u'} - \cancel{2e^x u}$$
$$+ \cancel{e^x u} = 2e^x + 8e^{-x}$$

So $e^x u'' = 2e^x + 8e^{-x}$

$$u'' = 2 + 8e^{-2x}$$

$$u' = 2x + \left(-\frac{8}{2}\right)e^{-2x} + c_1$$

$$u = x^2 - \frac{4}{-2}e^{-2x} + c_1 x + c_2$$
$$= x^2 + 2e^{-2x} + c_1 x + c_2$$

$$y = e^x u$$
$$= x^2 e^x + 2e^{-x} + c_1 x e^x + c_2 e^x$$ Same

Ex2 $y''+y=\sec x$

Guessing would be almost impossible

$y''+y=0 \qquad y_c=c_1\cos x+c_2\sin x$

Reduction of Order

$y=u\cos x$

$y'=u'\cos x-u\sin x, \qquad y''=u''\cos x-2u'\sin x-u\cos x$

Sub $y''+y=\sec x$

$u''\cos x-2u'\sin x-\cancel{u\cos x}+\cancel{u\cos x}=\sec x$

linear

let $u'=v$ so $u''=v' \Rightarrow \quad \cos x\frac{dv}{dx}-2\sin x\, v=\frac{1}{\cos x}$

$$\frac{dv}{dx}-\frac{2\sin x}{\cos x}v=\frac{1}{\cos^2 x} \qquad \mu=e^{\int\frac{-2\sin x}{\cos x}dx}=e^{2\ln\cos x}=\cos^2 x$$

$$\Rightarrow \frac{d}{dx}(\cos^2 x\, v)=\cos^2 x\cdot\frac{1}{\cos^2 x}=1$$

$$\Rightarrow \cos^2 x\, v=x+c_1 \qquad v=x\sec^2 x+c_1\sec^2 x \quad \left(v=\frac{du}{dx}\right)$$

$$u=\int x\sec^2 x+c_1\tan x+c_2$$

integration by parts

$u = x \qquad v = \tan x$

$du = dx \qquad v = \sec^2 x\, dx$

$$= x\tan x - \int \tan x\, dx = x\tan x + \ln(\cos x) + C$$

$$u = x\tan x + \ln|\cos x| + c_1 \tan x + c_2$$

$y = \cos x \; u$

$$= \underbrace{x\sin x + \cos x \ln|\cos x|}_{y_p} + \underbrace{c_1 \sin x + c_2 \cos x}_{y_c}$$

So the general sol^n is

$$y = c_1 \sin x + c_2 \cos x + x\sin x + \cos x \ln|\cos x|$$