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Research Article

Certain Properties of Hilbert Space Operators

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Abstract

Let H be an infinite dimensional Hilbert space and $B(H)$ the algebra of all bounded linear operators in H. In this paper, we characterize certain properties of operators on Hilbert spaces. These include: norm and singular value inequalities. We have proved that if A has closed range and the Moore-Penrose inverse A^+ and if we let X vary in $NA(H)$, where A A* AX=A*. Then A^+ belongs to $NA(H)$ and $s_i(X) \geq s_i(A^+)$ for $i = 1, 2, \ldots$.We have given extensions to other classes for example the norm-attainable class.

Keywords: Hilbert space, Norm, Inequalities, Norm-attainability.

Introduction

Studies of norm inequalities have been done over decades with interesting results obtained [1]. A unitarily invariant norm as an example is any norm defined on some two-sided ideal of $B(H)$ and $B(H)$ itself which satisfies the following two conditions. For unitary operators $U, V \in B(H)$ the equality $|||UXV||| = |||X|||$ holds, and $|||X||| = s_1(X)$, for all rank one operators X. It is proved [2] that any unitarily invariant norm depends only on the sequence of singular values. Also, it is known that the maximal ideal, on which $|||UXV|||$ has sense, is a Banach space with respect to that unitarily invariant norm. Among all unitarily invariant norms there are few important special cases [3]. We have several examples but the first is the Schatten p-norm ($p \geq 1$) defined by: $\|X\|_p = (\Sigma^{+\infty}_{j=1} s_j(X)^p)^{1/p}$ on the set $Cp = \{X \in B(H) : \|X\|p < +\infty\}$ [4].

For p = 1, 2 this norm is known as the nuclear norm (Hilbert-Schmidt norm) and the corresponding ideal is known as the ideal of nuclear (Hilbert-Schmidt) operators [5, 6]. The ideal C_2 is also interesting for another reason. Namely, it is a Hilbert space with respect to the $\|.\|_2$ norm [7]. The other important special case is the set of so-called KyFan norms [8] given by $\|X\|_k = \Sigma^{k}_{j=1} s_j(X)$. The well-known KyFan [9] dominance property asserts that the condition $\|X\|k \leq \|Y\|$ for all $k \geq 11$ [10-18]. In this paper, we characterize certain properties of operators on Hilbert spaces. These include: norm inequalities, singular value inequalities. Present paper gives extensions to other classes for example the norm-attainable class.

Materials and methods

In the present work, H is a complex, separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$; and $B(H)$, the set of all bounded, linear operators mapping H to H. An operator A is said to be norm-attainable if there exist a unit vector xi in H such that $\|Ax\| = \|A\|$. $NA(H)$ is the class of all norm-attainable operators on H. An operator A is self-adjoint if $A = A^*$; equivalently, if $< Af, f > \in R$ for all f in H; a (self-adjoint) operator is positive, denoted $A \geq 0$, if $\langle Af, f \rangle \geq 0$ for all f in H [19]. The notation $A \geq B$, for self-adjoint A and B, means that $A - B \geq 0$. It can be shown that every positive operator T has a unique positive square root, denoted $T^{\frac{1}{2}}$. The modulus $|A|$ of an arbitrary operator A is the positive square root of (the positive operator) A^*A; that is, $|A| = (A^*A)^{\frac{1}{2}}$. Recall that an operator A^- is a generalized inverse of A if $AA^-A = A$. An operator A has a generalized inverse if and only if its range, Ran A, is closed (We define Ran $A = \{Af : f \in H\}$ [12, Theorem 12.9]). For an operator A, with closed range, its Moore-Penrose inverse A^+ satisfies eq. (1).

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$$\begin{array}{lll} AA^+A = A & \text{(i)} & \\ A^+AA^+ = A^+ & \text{(ii)} & \\ (AA^+)^* = AA^+ & \text{(iii)} & (1) \\ (A^+A)^* = A^+A & \text{(iv)} & \end{array}$$

and, further, A^+ is uniquely determined by these properties [9, Theorem 1]. For the construction of the Moore-Penrose inverse of an operator with closed range [11]. If an operator A^- satisfies (i) and (iii) of (2.1) (so that $AA^-A = A$ and $(AA^-)^* = AA^-$) it will be called a (i), (iii) inverse of *A*; if A^- satisfies (i), (iv) of (2.1) it will be called a (i), (iv) inverse of *A*. Observe that if A^- is a (i), (iii) inverse of *A* then AA^- is the projection onto $\operatorname{Ran} A$ and that if A^- is a (i), (iv) inverse of *A* then A^-A is the projection onto $(\operatorname{Ker} A)^\perp$. An operator *A* is of finite rank *n*, denoted $\operatorname{rank} A = n$, if $\dim \operatorname{Ran} A = n < \infty$. The rank 1 operator *x* a $\langle x, f\rangle g$, for fixed vectors *f* and *g* in *H*, is denoted $f \otimes g$. The spectral theorem for compact, positive operators says that every compact, positive operator *X* can be expressed uniquely as eq. (2). $X = \sum_i \alpha_i (f_i \otimes f_i)$ (2)

where (α_i) is the sequence of positive eigenvalues of *X* arranged in decreasing order and repeated according to multiplicity, and (f_i) is the corresponding orthonormal sequence of eigenvectors (so that $S\{f_i\} = \operatorname{Ran} X$); and *X* is of finite rank *n* if and only if the sequence (α_i) terminates after just *n* terms [11, Theorem 1.9.3]. In particular, if $X = |A|$, for some *A* in *B(H)*, the eigenvalues α_i are called the singular values of *A* and denoted $s_i(A)$.

The spectral theorem for compact operators says that every compact operator *X* can be expressed uniquely as eq. (3). $X = \sum_i s_i(X)(f_i \otimes g_i)$ (3) where (f_i) and (g_i) are orthonormal sequences in *H* and $(s_i(X))$ is the sequence of singular values of *X*, arranged in decreasing order and repeated according to multiplicity; and *X* is of finite rank *n* if and only if the sequence $(s_i(X))$ terminates after just *n* terms [11, Theorem 1.9.3]. For a compact operator *A*, let $s_1(A)$, $s_2(A)$, … denote the singular values of *A*, arranged in decreasing order and repeated according to multiplicity. If, for some $p > 0$, $\sum_{i=1}^{\infty} s_i^p(A) < \infty$ we say *A* is in the (2.1) von Neumann-Schatten class C_p and write

$$\|A\|_p = \left[\sum_{i=1}^{\infty} s_i^p(A)\right]^{1/p}.$$

For $1 \le p < \infty$, it can be shown that $\|\cdot\|_p$ is a norm called the von Neumann-Schatten norm. We sometimes write $\|\cdot\|_\infty = \|\cdot\|$, the supremum norm. For all *p*, where $0 < p < \infty$, C_p is a 2-sided ideal of *B(H)* and for $1 \le p < \infty$ the space C_p is a Banach space under $\|\cdot\|_p$. For more details of the von Neumann-Schatten classes and norms see [2, Chapter XI], [11, Chapter 2]. The class C_1 is called the trace class. If $A \in C_1$ and (ϕ_i) is an orthonormal basis of *H* then the quantity $\operatorname{tr} A$, called the trace of *A* and defined by $\operatorname{tr} A = \sum_i \langle A\phi_i, \phi_i\rangle$, can be shown to be finite and independent of the particular basis (ϕ_i) chosen. Further, $A \in C_p$, where $1 \le p < \infty$, if and only if $|A| \in C_1$.From (2.3) it follows that for every *X* in C_p, where $1 \le p < \infty$, there exists a sequence (X_n) of operators, each of finite rank *n*, such that

$$\|X - X_n\|_p \to 0 \quad \text{as } n \to \infty. \qquad (4)$$

The von Neumann-Schatten norms $\|\cdot\|_p$, for $1 \le p < \infty$, satisfy the property of uniformity [2, Chap XI, 9.9(d)]: if, A, B is in B(H), and $X \in C_p$ then $\|AXB\|_p \le \|A\| \|X\|_p \|B\|$. (5)
We cite further results we shall need about the von Neumann-Schatten norms.

Proposition 2.1 [6, Lemma 3.1]. (a) If $|A|^2 \ge |B|^2$ then $\|A\| \ge \|B\|$;
(b) if, further, *A* is compact then $|B|$ is compact and $|A| \ge |B|$;
(c) if, further, $A \in C_p$, for $1 \le p < \infty$, then $B \in C_p$ and $\|A\|_p \ge \|B\|_p$.
Note that the polar decomposition [3, Chapter 16] says that every operator *A* can be expressed uniquely as $A = U|A|$ where the partial isometry *U* is such that $\operatorname{Ker} U = \operatorname{Ker}|A|$ (*A* partial isometry *U* satisfies $\|Uf\| = \|f\|$ for all *f* in $(\operatorname{Ker} U)^\perp$).

Lemma 2.2 [1, Theorem 2.1]. If $1 < p < \infty$, the map X a $\| X \|_p^p$ (from C_p to ¡ $^+$) is differentiable with derivative D_X at X given by $D_X(S) = p\,\mathrm{Re}\,\mathrm{tr}[|X|^{p-1} U^* S]$ where $X = U|X|$ is the polar decomposition of X. If the underlying space H is finite-dimensional the same result holds for $0 < p < \infty$ at every invertible element X.

Lemma 2.3 [6, Lemma 2.5]. If S is a convex set of operators in C_p, where $1 < p < \infty$, there is at most one minimizer of $\| X \|_p^p$ where $X \in S$.

A norm $|||\cdot|||$ is said to be unitarily invariant if $|||AU||| = |||VA|||$ for all unitary operators U and V (provided $|||A||| < \infty$). Examples of unitarily invariant norms are the supremum norm $\|\cdot\|$ and the von Neumann-Schatten norms $\|\cdot\|_p$, $1 \le p < \infty$. A unitarily invariant norm $|||\cdot|||$ is strictly convex if $|||A+B||| = |||A||| + |||B||| \Rightarrow A = kB$ for real, positive k (provided $|||A||| < \infty$ and $|||B||| < \infty$). The next result – a generalization of Lemma 2.2 – is proved similarly to it.

Lemma 2.4. Let $|||\cdot|||$ be a strictly convex, unitarily invariant norm and let $\{X \text{ in } L(H) : |||X||| < \infty\} = \Im$, say. Then if S is a convex set of operators in $\Im$ there is at most one minimizer of $|||X|||$, where $X \in S$. Many properties of unitarily invariant norms can be deduced via Lemma 2.4 from those of symmetric gauge functions provided the operators concerned are of finite rank. References below are to Schatten's own elegant exposition [11]. Let F be the set of all operators of finite rank and let L be the set of all sequences of real numbers having a finite number of non-zero terms. A symmetric gauge function $\phi : L \to R+$ is a function satisfying the following properties:

$$\phi(u) > 0 \quad \text{if } u \ne 0 \qquad \text{(i)}$$
$$\phi(\alpha u) = |\alpha|\,\phi(u) \quad \alpha \in R \qquad \text{(ii)}$$
$$\phi(u+v) \le \phi(u) + \phi(v) \qquad \text{(iii) (6)}$$
$$\phi(u_1, ..., u_n) = \phi(e_1 u_{i_1}, ..., e_n u_{i_n}) \qquad \text{(iv)}$$

where each $e_i = \pm 1$ and $\{i_1, ..., i_n\}$ is a permutation of $\{1, ..., n\}$. (In (2.6) (iv), and below, we write $\phi(u_1, u_2, ...)$ instead of $\phi(u)$ for the values of the symmetric gauge function ϕ).

Proposition 2.5 [12, Lemma 6]. Let ϕ be a symmetric gauge function. If $u_i \ge v_i \ge 0$ for $1 \le i \le n$ then $\phi(u_1, ..., u_n, 0, ...) \ge \phi(v_1, ..., v_n, 0, ...)$.

Theorem 2.6 [12, Theorem 8]. Let A be in F and let $s_1(A), s_2(A), ...$ be the singular values of A arranged in decreasing order and repeated according to multiplicity. If $\phi : L$ α $R+$ is a symmetric gauge function then the function $\Phi : F$ α $R+$, $\Phi(A) = \phi(s_1(A), s_2(A), ...)$ vis a unitarily invariant norm $|||\cdot||| : F$ a ¡ $^+$, i.e., $|||A||| = \Phi(A)$; and, conversely, if $|||\,.\,||| : F$ α $R+$ is a unitarily invariant norm then there exists a symmetric gauge function ϕ such that $|||A||| = \phi(s_1(A), s_2(A), ...)$.

It follows from Theorem 2.6, first, that every unitarily invariant (ui) norm $|||\cdot|||$ is self-adjoint, i.e., $|||A||| = |||A^*|||$ (because the non-zero eigenvalues of $(A^*A)^{1/2}$ and $(AA^*)^{1/2}$ are equal); and, second, that every unitarily invariant norm $|||\cdot|||$ has the property of uniformity: if A, B belongs to B(H) and $X \in F$ then [12, p.71, Theorem 11] $|||AXB||| \le \|A\|\,|||X|||\,\|B\|$. (7).

Results and discussion

Proposition 3.1. Let A be in C_p, where $1 \le p < \infty$, and B such that $|A|^2 \ge |B|^2$. Then:

(a) $B \in C_p$ and $s_i(A) \ge s_i(B)$ for $i = 1, 2, ...$;

(b) If $\mathrm{rank}\, A = n < \infty$ then $\mathrm{rank}\, B \le n$ and for every ui norm $|||\cdot||| : F$ a ¡ $^+$, $|||A||| \ge |||B|||$.

Proof. The proof takes two cases. The first case is a special one. Suppose, first, that $\mathrm{rank}\, A = n < \infty$. Then $\mathrm{rank}\,|A| = n$ and, as a consequence of (2), $|A| = \sum_{i=1}^{n} \alpha_i (f_i \otimes f_i)$ where $\alpha_i > 0$, $|A| f_i = \alpha_i f_i$ and $S\{f_i\} = \mathrm{Ran}\,|A|$ (= $(\mathrm{Ker}\,|A|)^{\perp}$). As $|A|$ is compact and $|A|^2 \ge |B|^2$ it follows from Lemma 2.1 (b) that (2.6) $|B|$ is compact. Therefore, $|B| = \sum_{i=1}^{\infty} \beta_i (g_i \otimes g_i)$ where $\beta_i > 0$, $|B| g_i = \beta_i g_i$ and $S\{g_i\} = \mathrm{Ran}\,|B|$ $(= (\mathrm{Ker}\,|B|)^{\perp}$. We prove that $\mathrm{Ran}\,|A| \supseteq \mathrm{Ran}\,|B|$; equivalently, $\mathrm{Ker}\,|A| \subseteq \mathrm{Ker}\,|B|$. Let f be in $\mathrm{Ker}\,|A|$. Then, since by Lemma 2.1 (b), $|A|\geq|B|$ we have $0\leq\langle|B|f,f\rangle\leq\langle|A|f,f\rangle=0$ so that $f\in\text{Ker}|B|$. Hence, $\text{Ker}|A|\subseteq\text{Ker}|B|$. Therefore, $\text{rank}|B|\leq n$ and so $\text{rank}B\leq n$. Since $|A|$ and $|B|$ are of finite rank and since $\text{Ran}X=(\text{Ker}X)^{\perp}$ for $X=|A|,|B|$ we can apply Lőwner's result [8, p.510, A1b] that says that if *S* and *T* are positive $n\times n$ matrices such that $S\geq T$ then $\lambda_i(S)\geq\lambda_i(T)$ (here, $\lambda_i(S)$ denotes the i^{th} (positive) eigenvalue of *S*, the eigenvalues being arranged in decreasing order and repeated according to multiplicity). Hence, $\lambda_i(|A|)\geq\lambda_i(|B|)$, that is, $s_i(A)\geq s_i(B)$ for $i=1,2,...$ Next we prove a general case. We now extend this result to the von Neumann-Schatten classes. Let now *A* be in C_p, where $1\leq p<\infty$. Then, since $|A|^2\geq|B|^2$, it follows from Lemma 2.1 (c), (b) that $B\in C_p$ and $|A|\geq|B|$. As $A\in C_p$ then $|A|\in C_p$. Thus, $|A|-|B|$ is a positive operator in C_p and so by (2.2) can be expressed as $|A|-|B|=\sum_{i=1}^{\infty}\gamma_i(h_i\otimes h_i)$ where $\gamma_i>0$, $(|A|-|B|)h_i=\gamma_ih_i$ and $S\{h_i\}=\text{Ran}(|A|-|B|)$ $(=(\text{Ker}(|A|-|B|))^{\perp}$). Thus, for each fixed *n* $(<\infty)$ each $S\{h_i\}_{i=1}^{i=n}$ reduces $|A|-|B|$. Therefore, by the second paragraph of the proof of the special case $s_i(A)\geq s_i(B)$ for $i=1,....,n$. (8)

To extend this to all $i=1,2,...$ recall (cf. (2.4)) that the operator X $(=A,B)$ is the uniform limit in $\|\cdot\|_p$ of a sequence of finite rank operators X_n, where $\text{rank}X_n=n$. Thus,

$X=\sum_{i=1}^{\infty}s_i(X)(l_i\otimes m_i),\ X_n=\sum_{i=1}^{n}s_i(X)(l_i\otimes m_i)$ so that $X-X_n=\sum_{i=n+1}^{\infty}s_i(X)(l_i\otimes m_i)$ whence, for $1\leq p<\infty$, $\|X-X_n\|_p=\left[\sum_{i=n+1}^{\infty}s_i^{\,p}(X)\right]^{1/p}\to0$ as $n\to\infty$ and so $(s_i(X))_{i=1}^{n}\to(s_i(X))_1^{\infty}$. Taking X $(=A,B)$ the inequality (8) extends to all *i*.

(b) As in the Special case of (a), it follows that if $\text{rank}A=n<\infty$ then $\text{rank}B\leq n$. If ϕ is the symmetric gauge function associated, by Theorem 2.2, with the unitarily invariant norm $|||\cdot|||$ then, since $s_i(A)\geq s_i(B)$ for $i=1,...,n$, it follows from that $\phi(s_1(A),...,s_n(A))\geq\phi(s_1(B),...,s_n(B))$, that is, $|||A|||\geq|||B|||$. ■

Theorem 3.2 Let *A* have closed range and properties (i)-(iv) of inverse A^- and let *X* be such that $AX-C\in C_p$ where $1\leq p<\infty$. Then $s_i(AX-C)\geq s_i(AA^-C-C)$ for $i=1,2,\ldots$ (9);

and if $AX-C$ is of finite rank so is AA^-C-C and for every unitarily invariant norm $\|.\|:F\alpha\ R+$,

$|||AX-C|||\geq|||AA^-C-C|||$ (10)

with equality in (9) if, and for strictly convex $|||\cdot|||$ only if, $AX=AA^-C$, that is, $X=A^-C+(I-A^-A)L$ for arbitrary *L* in $L(H)$; if $A^-=A^+$ the least such minimizer of finite rank (with respect to $|||\cdot|||$) is A^+C.

Proof. As in [6, Theorem 3.3] the inequality $|AX-C|^2\geq|AA^-C-C|^2$ yields, by Propositio 3.1 (a), the inequality (3.2) and for finite rank $AX-C$ yields, by Proposition 3.1 (b), the inequality (9). Equality holds in (9) if $AX=AA^-C$, that is, if $X=A^-C+(I-A^-A)L$ for arbitrary *L*. If, further, $A^-=A^+$ the inequality $|A^+C+(I-A^+A)L|^2\geq|A^+C|^2$ of [6, Theorem 3.3] yields, by Proposition 3.1 (b), that A^+C is a least such minimizer with respect $|||\cdot|||$. For strictly convex $|||\cdot|||$ the equality assertions follow from Lemma 2.3 and the convexity of the sets $\{AX-C:X\text{ variable}\}$ and $\{A^+C+(I-A^+A)L:L\text{ variable}\}$. ■

We have a similar "left-handed" result pertaining to $XB-C$. This says, amongst other things, that if *B* has closed range and has a (i), (iv) inverse B^- and if $XB-C\in C_p$, where $1\leq p<\infty$, then $s_i(XB-C)\geq s_i(CB^-B-C)$ for $i=1,2,\ldots$. It might be thought that Proposition 3.1 is part of some "2-sided" result pertaining to $AXB-C$. Certainly, it is true that if *A* and *B* have closed ranges, if *A* has a (i), (iii) inverse A^-, if *B* has a (i), (iv) inverse B^- and if $AXB-C\in C_2$ then

$AA^{-}CB^{-}B-C\in C_2$ and $\|AXB-C\|_2\geq\|AA^{-}CB^{-}B-C\|_2$, [10, Corollary 1]. But it is not true that $s_i(AXB-C)\geq s_i(AA^{-}CB^{-}B-C)$ for $i=1,2,\ldots$.

Example 3.3 [7, Example 4.1]. Let $H=R\times R$ and $A=B=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ so that $A^{+}=B^{+}=\begin{pmatrix}1/4&1/4\\1/4&1/4\end{pmatrix}$. Take $X=\begin{pmatrix}1/4&1/4\\1/4&1/4\end{pmatrix}$ and $C=\begin{pmatrix}1&1\\1&5\end{pmatrix}$. Then it can be checked that $|AXB-C|^2=\begin{pmatrix}0&0\\0&16\end{pmatrix}$ and $|AA^{+}CB^{+}B-C|^2=\begin{pmatrix}2&-2\\-2&10\end{pmatrix}$ which gives

$$s_1(AXB-C)=4\geq(6+\sqrt{20})^{1/2}=s_1(AA^{+}CB^{+}B-C)$$

$$s_2(AXB-C)=0<(6-\sqrt{20})^{1/2}=s_2(AA^{+}CB^{+}B-C)$$

The next result contains and extends some already known inequalities [5, Theorems 3.1, 3.2], [6, Theorems 3.5, 3.6].

Theorem 3.4. Let A be fixed, let P and Q be fixed projections such that $A=PAQ$ and let X vary in C_p, where $1\leq p<\infty$, and be such that $A=PXQ$. Then $A\in C_p$ and

(a) $s_i(X)\geq s_i(A)$ fo $i=1,2,\ldots$;

(b) $\|X\|_p\geq\|A\|_p$ with equality occurring if, and for $1<p<\infty$ only if, $X=A$;

(c) V is a critical point of the map X a $\|X\|_p^p$, for $1<p<\infty$, if and only if $V=A$;

(d) if X is of finite rank then for every unitarily invariant norm $\|\!|.|\!\|: F\ \alpha\ R+$, $\|\!|X|\!\|\geq\|\!|A|\!\|$ with equality occurring if, and for strictly convex $\|\!|\cdot|\!\|$ only if, $X=A$.

Proof. (a) Obviously, since $X\in C_p$ then $A(=PXQ)\in C_p$. For a projection R, say, $|X|^2\geq|RX|^2$ (equivalently, $\|Xf\|\geq\|RXf\|$ where $f\in H$). Apply Proposition 3.1 (a) successively (for the projections P and Q)

$$s_i(X)\geq s_i(PX)=s_i(X^{*}P)\geq s_i(QX^{*}P)=s_i(PXQ)=s_i(A)$$

for $i=1,2,\ldots$ using the self-adjointness of $s_i(\cdot)$.

(b) The uniformity (5) of the $\|\cdot\|_p$ norm, for $1\leq p<\infty$, gives, since $\|P\|=\|Q\|=1$,

$$\|A\|_p=\|PXQ\|_p\leq\|P\|\,\|X\|_p\|Q\|=\|X\|_p.$$

For $1<p<\infty$, the uniqueness assertion follows, by Lemma 2.2 from the convexity of the set $\{X:PXQ=A\}$.

(c) Let V be a critical point of X a $\|X\|_p^p$, for $1<p<\infty$, and let S be an arbitrary increment of V so that $A=PVQ=P(V+S)Q$. Hence, $S\in C_p$ and $PSQ=0$. Take $S=f\otimes g$ for vectors f and g in H. Recall that $\mathrm{tr}[T(f\otimes g)]=\langle Tg,f\rangle$. Then by Proposition 2.1

$$0=\mathrm{Re}\,\mathrm{tr}[|V|^{p-1}U^{*}(f\otimes g)]=\mathrm{Re}\langle|V|^{p-1}U^{*}g,f\rangle \quad (11)$$

where $V=U|V|$ is the polar decomposition of V (so that $\mathrm{Ker}\,U=\mathrm{Ker}\,|V|$). As $0=PSQ=P(f\otimes g)Q$ then $f\in\mathrm{Ran}(I-Q)$ for arbitrary g or $g\in\mathrm{Ran}(I-P)$ for arbitrary *f*. Substituting $f=(I-Q)x$, for arbitrary *x* in (11) gives $0=\langle g,U|V|^{p-1}(I-Q)x\rangle$ which, since g is also arbitrary, forces $0=U|V|^{p-1}(I-Q)x$; hence, $\mathrm{Ran}\,|V|^{p-1}(I-Q)\subseteq\mathrm{Ker}\,U=\mathrm{Ker}\,|V|$ and so $\mathrm{Ran}(I-Q)\subseteq\mathrm{Ker}\,|V|^{p}=\mathrm{Ker}\,|V|\subseteq\mathrm{Ker}\,U|V|=\mathrm{Ker}\,V$. Hence, $V=VQ$. Similarly, substituting $g=(I-P)y$, for arbitrary *y*, in (3.5) yields $V=PV$. Thus: $V=PV=PVQ=A$ as desired. Conversely, by (b), A is a global minimizer of X a $\|X\|_p^p$ and hence, for $1<p<\infty$, is a critical point of it.

(d) As in (b), the inequality follows from the uniformity (2.7) of the $\|\!|\cdot|\!\|$ norm and the uniqueness assertion follows, via Lemma 2.3, from the convexity of the set $\{X:PXQ=A\}$. (Alternatively, the inequality follows, via Theorem 3.1 (b), from (a).) ■

There is no version of Theorem 3.2 (c) for $0<p<1$ in finite dimensions since if the critical point V and hence, by the proof of Theorem 3.2 (c), $A(=V)$ were invertible, and hence $1-1$ and onto, the condition $A=PAQ$ would force $P=Q=I$ and hence $X=A$ so that $\|X\|_p^p$ would be constant. For $P\neq I$ or $Q\neq I$ the hypotheses of Theorem 3.2 that $A=PXQ$ and $X\in C_p$, where $1\leq p<\infty$, force A and A^{*} to be finite rank. For suppose $P\neq I$; since $A=PXQ\in C_p$ then A is compact and has closed range and so is of finite rank. As A has closed range it has the Moore-Penrose inverse A^+; hence $A^* = (AA^+A)^* = A^*(AA^+)$ (by (2.1) (iii)) so A^* is also of finite rank (Of course, if $Q \neq I$ this is obvious from $A^* = QX^*P \in C_p$). Hence, in Theorem (3.2), *A* and A^* are of finite rank. In Theorems 3.5 and 3.6 the operator A^+, being compact and having closed range, is of finite rank; hence, the operators $A(= AA^+A)$ and $A^*(= A^*AA^+)$ are of finite rank. The next set of results are given in the norm-attainable class *NA(H).*

Theorem 3.5 Let the operators *A* and *B* be fixed, let *B* have closed range and have a (i), (iv) of inverse B^- and satisfy $A = B^-BA$ and let *X* vary in *NA(H),* where, and satisfy $A = B^-BX$. Then *A* is in *NA(H)* and $s_i(X) \geq s_i(A)$ for $i = 1, 2, \ldots$.

Proof. Immediate from Theorem 3.2 (a) on taking the projection $"P" = B^-B$ and $"Q" = I$. ■

Theorem 3.6 Let *A* have closed range and have the Moore-Penrose inverse A^+ and let *X* vary in *NA(H),* where, and $A^*AX = A^*$. Then A^+ belongs to *NA(H)* and $s_i(X) \geq s_i(A^+)$ for $i = 1, 2, \ldots$.

Proof. As shown in [6, Theorem 3.6] the hypothesis $A^*AX = A^*$ is equivalent to $A^+AX = A^+$. The result is now immediate from Theorem 3.2 (a) on taking $"P" = A^+A$, $"Q" = I$ and $"A" = A^+$. ■

Conclusions

In the present paper, we have characterized certain properties of operators on Hilbert spaces. These include: norm inequalities and singular value inequalities. We have also given extensions to other classes for example the norm-attainable class *NA(H).*

Conflicts of Interest

The authors declare no conflict of interest.

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