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Research Article

On Characterization of Normal Derivations

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Abstract

In the present paper we established orthogonality of normal derivations in C_p-Classes. We employ some results for normal derivations due to Mecheri, Hacene, Bounkhel and Anderson**.** Let T be a compact operator and φ be the subspace where T attains its norm, let φ_ϵ be the set of vectors from C_p-class which forms an angle less or equal to ϵ with φ. We show that two operators are orthogonal if their inner product is zero.

Keywords: Banach space; Hilbert space; Gateaux derivative; Orthogonality; Schatten-p Class.

Introduction

Studies on normal derivations have been done by a number of mathematicians and researchers especially their properties and interesting results have been obtained. In [1, 2] for instance, they give results on orthogonality of derivations in C_1 and C_∞ spaces. In this paper, we give results on orthogonality of normal derivations in C_p-classes. We have used polar decomposition [3], Direct sum decomposition [4] and inner product [5] to arrive at the main results. Given normal derivations, and T, a linear operator on a finite dimensional complex Hilbert space H, then for all commutator of the form AX -XA has trace 0 and therefore 0 is in the numerical range W(AX-XA). But, if H is infinite dimensional, then there exist bounded operators A and B on T such that W(AB-BA) is a vertical line segment in the open right half-plane as illustrated by [6].

The present paper initialized a study of a class D of operators A on H with the property that 0 belongs to W(AX-XA)˜ for all bounded operator X on H. Such operators are finite, since D contains all normal operators, all compact operators, all operators having a direct sum and of finite rank, and the entire C*-algebra generated by each of its members. In [7], they work on minimization of the C∞−norm of suitable affine mappings from *B*(*H*) to C∞ using Gˆ*a*teaux derivative. They consider mappings that generalize elementary operators and in particular the generalised derivations. The results obtained characterized global minima in terms of Birkhoff orthogonality.

In [8], they study orthogonality in C_1-classes and characterized all those operators S ∈ C1 ∩ Kerϕ which are orthogonal to Ran(ϕ|C_1) (the range of ϕ|C_1) when ϕ is one of the following elementary operators: EA,B : B(H) → B(H);EA,B(X) = Σn i=1A_iXB_i − X, the nuclear operator ΔA,B(X) = AXB−X, the inner derivation $\delta_{A,A}$ = AX−XA and ˜*EA,B : B(H)* → B(H); ˜EA,B(X) = Σn*i*=1A*i*XB_i, where (A_1, ...,A_n) and B_1, ...,B_n are n-tuples of bounded operators on *H*. Maher [9] showed that if *A* is normal and AT = TA, 1 ≤ *p* < ∞ and s ∈ kerδA,B∩C_p, then the map F_p defined by F_p(X) = ǁS−(AX−XA) $ǁ_p$ has a global minimizer at V if, and for 1 < *p* < ∞ only if, AV − V A = 0 i.e we have ǁS − (AX − XA) $ǁ_p$ ≥ ǁT$ǁ_p$, if , and for 1 < *p* < ∞ only if AV −V A = 0. In [10] they also worked on minimizing ǁT− (AXB−X) $ǁ_p$ and obtained a similar inequality as for [11], where the operator *AX* − *XA* is replaced by the operator ΔA,B(X) = AXB − X.

In [12], they proved that if kerΔA,B ⊆ kerΔA*,B* and T ∈ kerΔA,B ∩ C*p*, then the map F_p defined by F_p(*X*) = ǁT − (AXB − X) $ǁ^p_p$ has a global minimize at V if, and for 1 < *p* < ∞ only if AV B − V = 0. Hence F_p(*X*) = ǁT− (AXB − X) $ǁ^p_p$ ≥ ǁTǁ ^p_p, if , and for 1 < p < ∞ only if, AV B − V = 0. In addition, [13] proved that kerΔA,B ⊆kerΔA*,B* and T ∈ kerΔA,B ∩ C_P

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for $1 < P < \infty$, then the map F_p has a critical point at W if and only if $AWB-W = 0$ i.e if DWF_p is the Frechet derivative at W of F_p, the set $\{W \in B(H) : DWFp = 0\}$ coincides with $\ker\Delta A,B$(the Kernel of $\Delta A,B$).

Research methodology

In this section, we start by defining some key terms that are useful in the methodology.

Definition 2.1 ([14], Definition 1.2) A Banach space is a complete normed space.

Definition 2.2 ([15], Definition 33.1) A Hilbert space H is an inner product space which is complete under the norm induced by its inner product.

Definition 2.3 ([16], Definition 3.1) Let f be a function on an open subset U of a Banach space X into Banach space Y. f is Gateaux differentiable at $x \in U$ if there is bounded and linear operator $T: X \to Y$ such that $T_x(h) = \lim_{t\to 0} \frac{f(x+th)-f(x)}{t}$ for every $h \in X$. The operator T is called the Gateaux derivative of f at x.

Definition 2.4 ([6], Definition 0.1) Let X be a complex Banach space. Then $y \in X$ is orthogonal to $x \in X$ if for all complex λ there holds $\| x + \lambda y \| \geq \| x \|$.

Definition 2.5 ([11], Section 2) Let $T \in B(H)$ be compact. Then $s_1(T) \geq s_2(T) \geq \ldots \geq 0$ are the singular values of T i.e the eigenvalues of $\|T\| = (T^*T)^{\frac{1}{2}}$ counted according to multiplicity and arranged in descending order. For $1 \leq p \leq \infty$, $C_p = C_p(H)$ is the set of those compact $T \in B(H)$ with finite p-norm, $\| T \|_p = (\sum_{i=1}^{\infty} s_i(T)^p)^{\frac{1}{p}} = (tr|T|^p)^{\frac{1}{p}} < \infty$.

Results and discussion

In this section we give the main results. We establish orthogonality results of normal derivations in C_p-class.

Lemma 3.1 Let $S, T \in C_p$ and A be a normed ideal J such that $\ker\delta_S(A) \subseteq \ker\delta^*{}_S(A)$. Let $S|R|^{r-1}U^* = |S|^{r-1}U^*T$ where $r > 1$ and $R = U|R|$ is the polar decomposition of R, then $S|R|U^{-1} = |R|U^*T$ if and only if R is self adjoint and invertible.

Proof. If $A = |R|^{r-1}$ then we have $SAU^{-1} = AU^*T$. .... (1)

We prove that $SA^nU^{-1} = A^nU^*T$ ...(2) for all $n \geq 1$. If $R = U|R|$ then,

$\ker U = \ker|R| = \ker|R|^{r-1} = \ker A$ and $(\ker U)^{\perp} = (\ker A)^{\perp} = range\overline{(A)}$.

Hence the projection U^*U onto$(\ker A)^{\perp} = A$ and $AU^*UA = A^2$. Taking the adjoints of (2) and given $\ker\delta_S(A) \subseteq \ker\delta^*{}_S(A)$, we obtain that

$TU^*UA = UAS$

and

$SA^2SAU^*UA = AU^*TUA = AU^*U^*AS = A^2S.$

S commutes with the positive operator A^2 and hence it must commute with its square root i.e $SA = AS$. Suppose the map F_ψ has a local minimum on C_p at S, then $D_{\psi(S)}(\phi(T)) \geq 0$ for all $T \in C_p$ i.e $rRe\{tr(|\psi(S)|^{r-1}U^*\phi(T))\} \geq 0$ for all $T \in C_p$. This implies,

$Re\{tr(|\psi(S)|^{r-1}U^*\phi(T))\} \geq 0$ for all $T \in C_p$.

Let $f \oplus g$ be a rank one operator given by $x \mapsto \langle x, f\rangle h$ where f and h are arbitrary vectors in the Hilbert space H. Take $T = f \oplus g$,

then $tr(|\psi(S)|^{r-1}U^*\phi(T)) = tr(\phi^*(|\psi(S)|^{r-1}U^*)T)$.

Therefore $D_{\psi(S)}(\phi(T)) \geq 0$ is equivalent to $Re\{tr(\phi^*(|\psi(S)|^{r-1}U^*)T\} \geq 0$ for all $T \in C_p$ or equivalently $Re\langle\phi^*(|\psi(S)|^{r-1}U^*)h, f\rangle \geq 0$ for all $f, h \in H$. Choose $f = h$ such that $\|f\| = 1$, then

$Re\langle\phi^*(|\psi(S)|^{r-1}U^*)f, f\rangle \geq 0$ ... (3).

Let $f(x)$ be the map defined by $\sigma(A) \subset R^+$ by $f(x) = x^{\frac{1}{r-1}}$ for $1 < r < \infty$.

f is the uniform limit of a sequence of polynomials P_i without constant term as $f(0) = 0$, then inequality (3) implies $SP_i(A)U^* = P_i(A)U^*T$. Hence,

$SA^{\frac{1}{r-1}}U^* = U^*A^{\frac{1}{r-1}}T$.

Theorem 3.2 Let $S, T \in C_p$ be self adjoint and invertible such that $\ker\delta_{S,T} \subseteq \ker\delta_{S^*,T^*}$, then $A \in \ker\delta_{S,T} \cap ran\delta_{S,T}$ if and only if $\|A + \delta_{S,T}(X)\|_p \geq \|A\|_p$ for all $X \in C_p$.

Proof. Suppose $A \in \Delta_{S,T}$, then by (8, Theorem 3.4) we obtain

$\|A + \delta_{S,T}(X)\|_p \geq \|A\|_p$ for all $X \in C_p$.

Conversely, if $\|A + \delta_{S,T}(X)\|_p \geq \|A\|_p$ for all $X \in C_p$, then we have $S|A|U^{-1} = |R|U^*T$.

Given $\ker\delta_{S,T} \subseteq \ker\delta_{S^*,T^*}$, then

$T^*|A|^{r-1}U^{-1} = |A|^{r-1}U^*$.

Taking adjoints we obtain,

$SU|A|^{r-1} = U|A|^{r-1}T$.

From Lemma 3.1, it follows that $SU|A| = U|A|T$ and hence $A \in \ker\delta_{S,T}$.

Remark 3.3 The above theorem still holds if we consider $\Delta_{S,T}$ instead of $\delta_{S,T}$.

Lemma 3.4 Let T be a compact operator and φ be the subspace where T attains its norm, let φ_ϵ be the set of vectors from C_p-class which forms an angle less or equal to ϵ with φ. Further let S be self adjoint compact operator, such that $\|S\| \le \lambda$ where λ is a real number such that $\frac{2\lambda}{e_1(T)-e_2(T)-2\lambda} \le \tan\epsilon$. Then there holds, $\|T+S\| = max_{h\in\varphi,\|h\|=1}\langle (T+S)h,h\rangle$.

Proof: Consider a unit vector y represented by $= h + f$, where $h \in \varphi$ and $f \in \varphi^\perp$ then $y \in \varphi_\epsilon$ if and only if $\frac{\|f\|}{\|h\|} \le \tan\epsilon$. Also,
$e_2(T) = max_{h\in\varphi^\perp,\|h\|=1}\langle Th,h\rangle$.
Given that the operator $T+S$ is compact and selfadjoint, there exists a unit vector y such that $\langle (T+S)y,y\rangle = \|T+S\|$.
Since $y = h + f$, then
$\langle (T+S)y,y\rangle = \langle (T+S)h,h\rangle$
$+2Re\langle Sh,f\rangle + \langle (T+S)f,f\rangle$
However, since $|\langle (T+S)h,h\rangle| \le \|T+S\|\|h\|^2$, $|\langle Sh,f\rangle| \le \lambda\|h\|\|f\|$,
$|\langle (T+S)f,f\rangle| \le (e_1-\lambda)\|f\|^2$, we have
$\|T+S\| \le \|T+S\|\|h\|^2 +$
$2\lambda\|h\|\|f\| + (e_2+\lambda)\|f\|^2$
i.e i.e $(e_1-\lambda)\|f\| \le \|T+S\|\|f\| \le$
$2\lambda\|h\| + (e_2+\lambda)\|f\|$, taking care of $\|T+S\| \ge e_1-\lambda$, respectively
$(e_1-e_2-2\lambda)\|f\| \le 2\lambda\|h\|$ from which we conclude that $y \in \varphi_\epsilon$.

Theorem 3.5 Let $S,T \in C_p$. Then we have,

$$lim_{\rho\to 0^+}\frac{\|S+\rho T\|_{C_p}-\|S\|_{C_p}}{\rho} =$$
$$max_{h\in\varphi,\|h\|=1}Re\langle U^*Th,h\rangle,$$

where $S = U|S|$, and φ is the characteristic subspace of the operator $|S|$ with respect to its eigenvalue e_1 .

Proof. We have

$$lim_{\rho\to 0^+}\frac{\|S+\rho T\|_{C_p}-\|S\|_{C_p}}{\rho} =$$
$$lim_{\rho\to 0^+}\frac{\|(S^*+\rho T^*)(S+\rho T)\|^{\frac{1}{2}}-\|S\|}{\rho}$$
$$= lim_{\rho\to 0^+}\frac{\|S^*S+\rho(S^*T+T^*S)+\rho^2T^*T\|-\|S\|^2}{\rho(\|S^*S+\rho(S^*T+T^*S)+\rho^2T^*T\|^{\frac{1}{2}}+\|S\|)}$$

The denominator
$$\rho\left(\|S^*S+\rho(S^*T+T^*S)+\rho^2T^*T\|^{\frac{1}{2}}+\|S\|\right)$$
tends to $2\|S\|$. We consider the limit of the numerator. The operator S^*S and $\rho(S^*T+T^*S)+\rho^2T^*T$ satisfy the assumption of Lemma 3.3, for an arbitrary $\epsilon > 0$ and for corresponding small enough ρ, we obtain

$$lim_{\rho\to 0^+}\frac{\|S^*S+\rho(S^*T+T^*S)+\rho^2T^*T\|-e_1^2}{\rho} =$$
$$lim_{\rho\to 0^+}\frac{max_{h\in\varphi,\|h\|=1}\langle (S^*S+\rho(S^*T+T^*S)+\rho^2T^*T)h,h\rangle-e_1^2}{\rho} \le$$
$$lim_{\rho\to 0^+}max_{h\in\varphi,\|h\|=1}[\langle (S^*T+T^*S)h,h\rangle +$$
$$\rho\langle T^*Th,h\rangle] = max_{h\in\varphi,\|h\|=1}2Re\langle Th,Sh\rangle$$

n the other hand,

$$lim_{\rho\to 0^+}\frac{\|S^*S+\rho(S^*T+T^*S)+\rho^2T^*T\|-e_1^2}{\rho} =$$
$$lim_{\rho\to 0^+}\frac{max_{h\in\varphi,\|h\|=1}\langle (S^*S+\rho(S^*T+T^*S)+\rho^2T^*T)h,h\rangle-e_1^2}{\rho}$$

such that for all $\epsilon > 0$ one obtains

$$\frac{1}{\|S\|}max_{h\in\varphi,\|h\|=1}Re\langle Th,Sh\rangle \le$$
$$lim_{\rho\to 0^+}\frac{\|S+\rho T\|_{C_p}-\|S\|_{C_p}}{\rho} \le$$
$$\frac{1}{\|S\|}max_{h\in\varphi,\|h\|=1}Re\langle Th,Sh\rangle$$

We note that
$$inf_{\epsilon>0}max_{h\in\varphi,\|h\|=1}Re\langle Th,Sh\rangle =$$
$$max_{h\in\varphi,\|h\|=1}Re\langle Th,Sh\rangle$$
given T and S are in the sphere that is metric. Hence, we arrive at the result by taking the infimum over all ϵ, since for all $h \in \varphi$ there holds $h = \|S\|Uh$.

Conclusions

These results are on orthogonality of normal derivations in C_p-classes. It would be interesting to establish orthogonality of normal derivations in B(H)-the algebra of all bounded linear operators on a complex separable and infinite dimensional Hilbert space H. Much research has been on numerical range and numerical radius, whereby a lot of interest has been on characterization of both real and complex operators. Researchers have proven that if a normal operator has a closed numerical range then the extreme points of its numerical range are its singular values. Research has also been done on the numerical range of normal operators on a Hilbert space and therefore it would be interesting to study on the numerical range of posinormal, hyponormal, M-hyponormal and subnormal operators. It has also been showed that the numerical range of every bounded linear operator is convex. It would be interesting to consider this property for normal derivations.

Conflicts of Interest

Authors declare no conflict of interest.

References

[1] Anderson A. On normal derivations. Proceedings of American Mathematical Society 1979;38:129-140.

[2] Bounkhel M. On Minimizing the norm of Linear maps in C_p-classes. Applied Sciences 2006;8:40-47.

[3] Charalambos Aliprantis D, Owen B. Principles of Real Analysis. Academic Press, 3rd, 1998.

[4] Christopher H. Banach and Hilbert Space Review. Lecture Notes 2016:1-13.

[5] Daryoush B, Encyeh DN. Introduction of Frechet and Gateaux Derivative. Applied Mathematical Sciences 2008;2:975-980.

[6] Keckic D. Orthogonality in C_1 and C_∞ spaces and normal derivations. Journal of Operator Theory 2014;51:89-104.

[7] Kittaneh F. Normal derivations in normal ideals. Proceedings of American Mathematical Society 2005;123:1779-1785.

[8] Mecheri S. Some recent results on operator commutators and related operators with applications, Lecture Monograph, 2010.

[9] Mecheri S, Hacene M. Gateaux Derivative and Orthogonality in C_∞, Journal of inequalities in Pure and Applied Mathematics., 20(2012), 275-284.

[10] Mecheri S, Bounkhel M. Global minimum and Orthogonality inC_1-classes. Journal of Mathematical Analysis and Applications Math 2013;1:51-60.

[11] Mecheri S. The Gateaux Derivative Orthogonality in C_∞, Lecture Notes, 2011.

[12] Vijayabalaji S, Shyamsundar G. Interval-valued intuitionistic fuzzy transition matrices. International Journal of Modern Science and Technology 2016;1(2):47-51.

[13] Judith JO, Okelo NB, Roy K, Onyango T. Numerical Solutions of Mathematical Model on Effects of Biological Control on Cereal Aphid Population Dynamics. International Journal of Modern Science and Technology 2016;1(4):138-143.

[14] Judith JO, Okelo NB, Roy K, Onyango T. Construction and Qualitative Analysis of Mathematical Model for Biological Control on Cereal Aphid Population Dynamics. International Journal of Modern Science and Technology 2016;1(5):150-158.

[15] Vijayabalaji S, Sathiyaseelan N. Interval-Valued Product Fuzzy Soft Matrices and its Application in Decision Making. International Journal of Modern Science and Technology 2016;1(7):159-163.

[16] Chinnadurai V, Bharathivelan K. Cubic Ideals in Near Subtraction Semigroups. International Journal of Modern Science and Technology 2016;1(8):276-282.

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