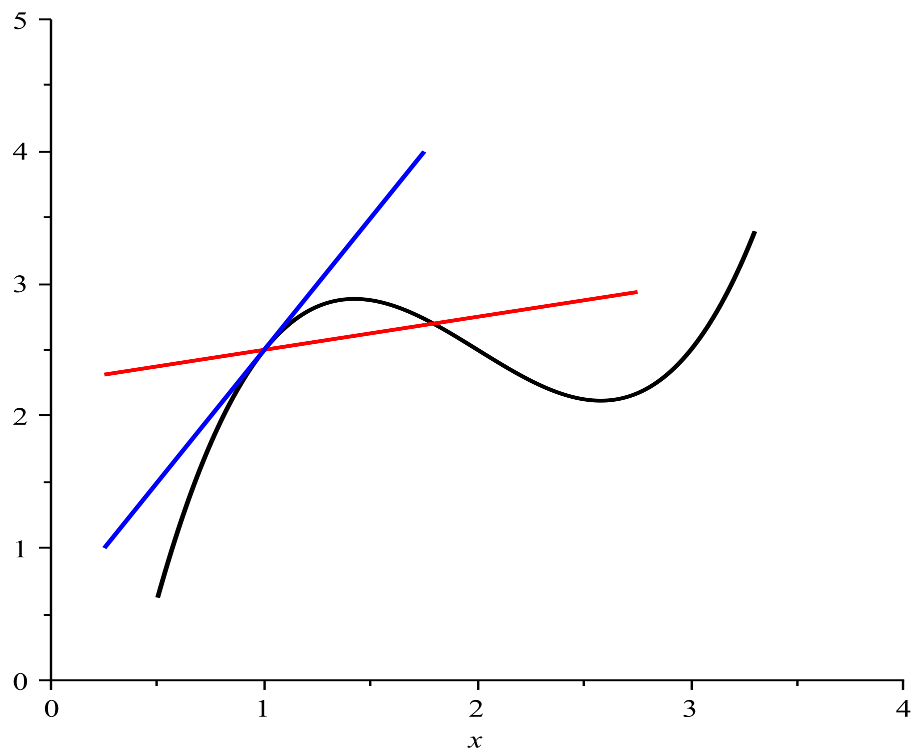
Calculus 3 - Tangent Plane

In calculus 1 we introduced the derivative. We considered the function $y = f(x)$ and a secant to the curve that goes through the points $(x, f(x))$ and $(x+h, f(x+h))$ (in red). Then we let $h \to 0$ and the secant line (red) becomes the tangent line (blue)



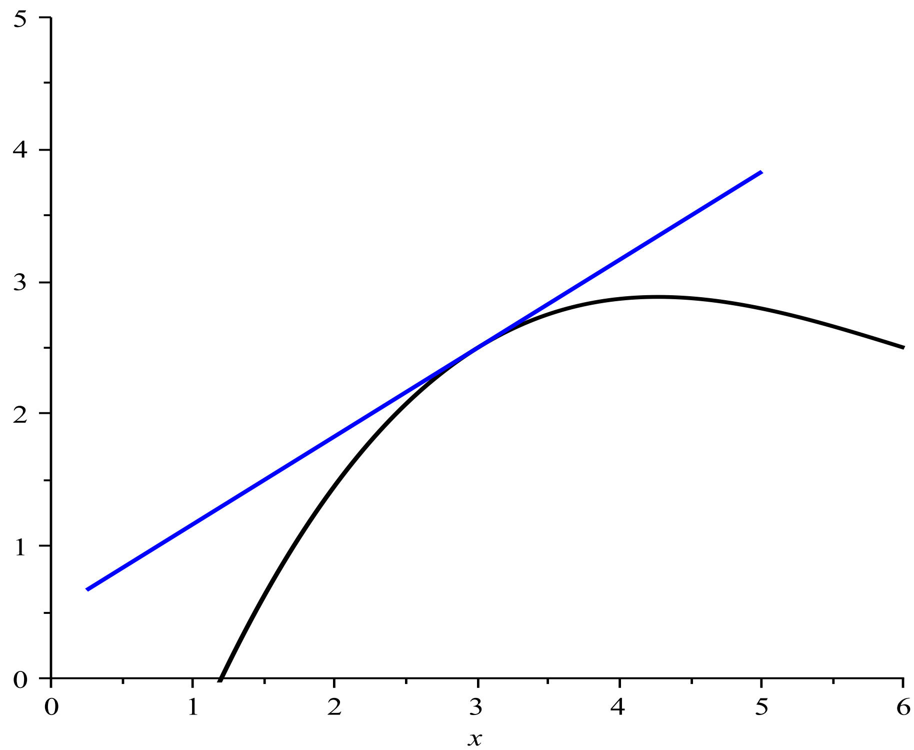
Mathematically, we define the derivative of a function $y = f(x)$ as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

As the derivative is the slope of the tangent, we were able to define the equation of the tangent at some point, say $x = a$ as

$$y - f(a) = f'(a)(x-a). \tag{2}$$

In analogy to tangent lines in 2D we have tangent planes in 3D. To get an idea on how to derive the equation of the tangent plane let us return to the tangent line problem.



We will associate a vector with the tangent and in particular the vector

$$\vec{u} = < 1, f'(x) > \tag{3}$$

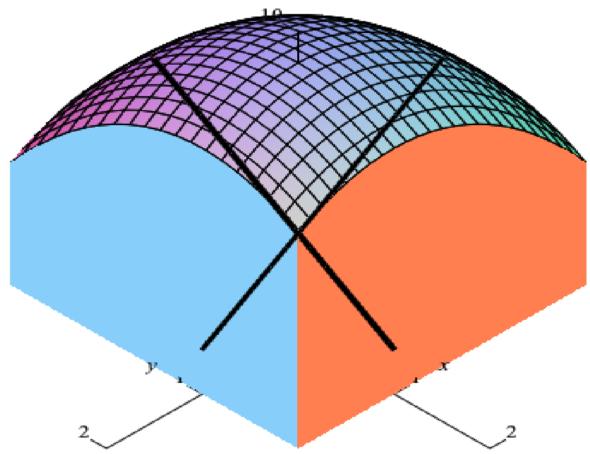
We then evaluate this vector at some point $x = a$ so

$$\vec{u} = < 1, f'(a) > \tag{4}$$

From the part of equations of lines, using the vector (4) and a point on the line $(a, f(a))$ we would obtain

$$x = a + 1t, \quad y = f(a) + f'(a) \cdot t \tag{5}$$

and eliminating t gives (2). Now we return to the partial derivatives we obtain last class. Recall we obtain two tangent lines by holding one variable fixed and varying the other.



So now we associate two vectors with these two tangent lines. If the surface is $z = f(x,y)$ then the tangent vectors are

$$\vec{u} = < 1, 0, f_x >, \quad \vec{v} = < 0, 1, f_y > . \tag{6}$$

We evaluate these at some point (a,b). We now cross these two vectors to get the normal so

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{vmatrix} = < -f_x(a,b), -f_y(a,b), 1 > . \tag{7}$$

The equation of the tangent plane is then

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + (z-c) = 0 \tag{8}$$

or

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z-c) = 0 \tag{9}$$

where $c = f(a,b)$.

Example 1. Find the equation of the tangent plane of $z = x^2 + y^2$ at $(1,2)$

Soln. We first find the partial derivatives so

$$f_x = 2x, \quad f_y = 2y. \tag{10}$$

We evaluate these at the point $(1,2)$ so

$$f_x = 2, \quad f_y = 4. \tag{11}$$

We also see that $c = f(1,2) = 5$. From (9) we have

$$2(x-1) + 4(y-2) - (z-5) = 0 \tag{12}$$

or

$$2x + 4y - z = 5. \tag{13}$$

Example 2. Find the equation of the tangent plane of $z = \sin(xy)$ at $(1, \pi/4)$

Soln. We first find the partial derivatives so

$$f_x = \cos(xy) \cdot y, \quad f_y = \cos(xy) \cdot x. \tag{14}$$

We evaluate these at the point $(1, \pi/4)$ so

$$f_x = \cos\left(\frac{\pi}{4}\right)\frac{\pi}{4} = \frac{\sqrt{2}\pi}{8}, \quad f_y = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}. \tag{15}$$

We also see that $c = f(1, \pi/4) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. From (9) we have

$$\frac{\sqrt{2}\pi}{8}\Big(x-1\Big) + \frac{\sqrt{2}}{2}\Big(y - \frac{\pi}{4}\Big) - \Big(z - \frac{\sqrt{2}}{2}\Big) = 0 \tag{16}$$

Example 3. Find the equation of the tangent plane of $x^2 + y^2 + z^2 = 9$ at $(2, 1, -2)$,

In this example z is given implicitly. Instead of trying to solve for z we use implicit differentiation. We create a function F and the derivatives are given as

$$z_x = -\frac{F_x}{F_z}, \quad z_y = -\frac{F_y}{F_z} \tag{17}$$

Let us pursue this in general first. The equation of the tangent plane is

$$-z_x(a,b)(x-a) - z_y(a,b)(y-b) + (z-c) = 0 \tag{18}$$

Now we use (17)

$$-\left(-\frac{F_x}{F_z}\right)(x-a) - \left(-\frac{F_y}{F_z}\right)(y-b) + (z-c) = 0 \tag{19}$$

or

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0 \tag{20}$$

So for example 3 we have

$$F_z = 2x, \quad F_y = 2y, \quad F_z = 2z \tag{21}$$

At the point $(2, 1, -2)$ we have

$$F_z = 4, \quad F_y = 2, \quad F_z = -4 \tag{22}$$

and from (20), the equation of the tangent plane is

$$4(x - 2) + 2(y - 1) - 4(z + 2) = 0 \tag{23}$$