Graphical Comparison of Convergence of Iterative Methods for Solution of Algebraic and Transcendental Equations

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***Abstract*:** In this paper, the graphical comparison between iterative methods has been shown. Motive for this project is to discern the best method of solving algebraic and transcendental equations by comparing different methods graphically. Cost of solving algebraic and transcendental equations is controlled by both, per iteration cost and required number of iterations. Derivations of all three methods were acquired. An example is demonstrated using MATLAB R2016a to show the results of all three methods and the results were cumulated, organized in tabular form and analyzed regarding their rate of convergence and errors in the solution. The results were also examined from graphical comparison of all three methods. It is concluded that the higher the rate of convergence, faster will be the evaluation of approximate solution of equation. Therefore, it was suggested that the Newton-Raphson method is the best method for solving algebraic and transcendental equations, in terms of a single variable, due to having higher rate of convergence among all three.

Keywords: *Iterative methods, convergence and divergence.*

AMS Subject Classification: *65F10, 65-01*

1. Introduction

Various useful solutions of the problems involving algebraic and transcendental equations are provided by Numerical methods. Algebraic equations may have infinite number of solutions and transcendental equations may have finite or infinite number of solutions or may not have a solution at all.

This research work will make emphasis on solving algebraic and transcendental equations in one dimension by including a single unknown, $F: R \to R$ which contains a scalar x as its solution, such that $f(x) = 0$.

It is necessary to make attempts to discuss about the branch of mathematics through which this work arises which is numerical analysis.

Numerical analysis is that subdivision or branch of mathematics which tackles with the development and use of numerical methods for solving problems. It is an area of mathematics and computer science that design, analyses and carry out algorithm for acquiring numerical solutions to problems involving continuous variables. Numerical algorithms are as ancient as the Egyptian Rhind Papyrus(1650 BC) which outlines a method of finding root for solving equations like algebraic and transcendental equations or in general, a problem in which calculating the values of unknowns $x_1, x_2 \dots x_n$ is required for which $f_k(x_1, x_2 \dots x_n) = 0$, $k = 1,2 \dots n$.

Algebraic and transcendental equations are those equations which never shows a straight line when we find their graphical solution.

Algebraic and transcendental equations are solved by using iterative methods. A random solution is assumed and is substituted into the equation and the error is calculated and then this error is used in a systematic manner to find an improved solution of the equation. Various methods are used to find the root of the equation.

Equations that can be transformed in the form of polynomial are referred as algebraic equations and the equations containing complex transcendental functions like trigonometric, hyperbolic, exponential or logarithmic functions are referred as transcendental equations. It is well known that when we find the convergence of several methods, the convergence of Newton method is guaranteed by quadratic rate.

If $f(x)$ is a polynomial of degree two or three or four, then formulae are available to find the root of polynomial. But, if $f(x)$ is a transcendental function like $a + be^x + \sin x + d \log x$ etc. the solution is not exact. Various numerical approximate methods can be used to solve such algebraic and transcendental equations. These types of equations are difficult to be solved so the best method to solve these equations is iterative methods. One of the best methods is Newton method which has second order rate of convergence. Many problems in science and engineering contain nonlinear and transcendental functions in the equation of the form $f(x)$. Numerical methods such as Newton method are generally used to find the approximate solution of such problems because it cannot be possible to find the exact root with the help of algebraic formulae.

1.1. Statement of Problem

Algebraic and transcendental equations are considered to be one of the most difficult equations in field of Science and by this reason this area gets less attention as compared to other forms of equations.

1.2. Aim and Objective of the Study

This study is aimed towards determining the method which is the best in solving algebraic and transcendental equations using iterative methods.

The main objectives of this study are as follows:

i. Determining if the solution exists or not and if the method converges to a single value or not.
ii. Comparing convergence of several methods numerically as well as graphically to determine the most suitable method for the solving the equation.

2. Materials and Methods

For reaching the objective of this work, following methods were compared: Regula Falsi method, bisection method and Newton method.

2.1. Bisection Method

Let us take an equation $f(x) = 0$ and we need to find the solution of this equation in the range (a, b). Also assume that $f(x)$ is continuous function and it can be either algebraic or transcendental. We know the fact that $\exists$ at least one root of the equation between a and b only if $f(a)$ and $f(b)$ are having opposite signs.

For the purpose of first approximation, let us assume that one root to be equal to $x_0 = \frac{a+b}{2}$, which is the midpoint of the range taken.

Then we will find the sign of $f(x_0)$ and check the conditions below-

i. If $f(x_0)$ is negative, the root of the equation will lie between x_0 and b.
ii. If $f(x_0)$ is positive, the root of the equation will lie between a and x_0.
iii. If $f(x_0) = 0$, the root of the equation will be equal to x_0.

Any of these conditions can be true.

Repeated bisection of the range of interval is required to find the solution of the equation and picking up the correct half of the range which also satisfies the sign conditions mentioned above. The relation $\left|\frac{b-a}{2^k}\right| \leq \epsilon$ can be used to find the number of iterations required in determining the root of the equation.

Hence we can conclude by above facts that the general formula for bisection method is-

$$x_k = \frac{a+b}{2}$$

<u>*Convergence of Bisection Method:*</u>

Consecutive approximations x_k of any root $x = \alpha$ of an equation $f(x) = 0$ is supposed to be convergent by an order $q \geq 1$ if,

$$|x_{k+1} - \alpha| \leq c|x_k - \alpha|$$

Where q, k and c are positive numbers. When $q = 1$ and $0 < c < 1$ then it is said to have first order convergence and constant c represents convergence rate.

2.2. Method of False position

This method is also known as Regula Falsi Method.

Take into consideration an equation $f(x) = 0$ and assume that sign of $f(a)$ and $f(b)$ are opposite and also assume $a < b$.

The point at which the curve $y = f(x)$ will meet the X-axis is between $A\ (a, f(a))$ and $B((b, f(b))$.These points will join together forming a chord whose equation is given by-

$$\frac{y - f(a)}{x - a} = \frac{f(a) - f(b)}{a - b}$$

An approximate value of the root of equation $f(x) = 0$ is given by the X-coordinate of the point of intersection of chord, putting $y = 0$ in the equation of chord, we get-

$$\frac{-f(a)}{x - a} = \frac{f(a) - f(b)}{a - b}$$

$$x[f(a) - f(b)] - af(a) + af(b) = -af(a) + bf(a)$$

$$x[f(a) - f(b)] = bf(a) - af(b)$$

Hence,

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

An approximate value of root of the equation $f(x) = 0$ is given by x_1 which is between a and b.

So $x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$.

Now sign of $f(x_1)$ and $f(a)$ are opposite to each other or sign of $f(x_1)$ and $f(b)$ are opposite.

Assume that $f(x_1).f(a) < 0$ then x_2 will lie in between $x_1$and a.

Therefore $x_2 = \frac{af(x_1)-x_1f(a)}{f(x_1)-f(a)}$

We can now find $x_3, x_4, \ldots$ in the similar way.

Hence the general formula will be-

$$x_{k+1} = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

<u>*Convergence of RegulaFalsi Method:*</u>

Let $\langle x_k \rangle$ be any sequence of approximate values which are obtained from

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} f(x_k)$$

And α be the precise value of root of the function $f(x) = 0$, then

$$x_{k+1} = \alpha + e_{k+1}$$

Here e_{k+1} is the error in $(k+1)^{th}$ term.

So,

$$\alpha + e_{k+1} = \alpha + e_k - \frac{(e_k - e_{k-1})}{f(\alpha + e_k) - f(\alpha + e_{k-1})} f(\alpha + e_k)$$

$$e_{k+1} = \frac{\frac{e_{k-1}}{2} f"(\alpha) + \cdots}{f'(\alpha) + \left[\frac{e_k + e_{k-1}}{2}\right] f"(\alpha) + \cdots}$$

And hence $f(x) = 0$.

2.3. Newton-Raphson Method

Newton method is a well-known and most popular numerical method and it is now introduced as most powerful method for solving algebraic and transcendental equations $f(x) = 0$. Let f' is assumed as the continuous derivative of f.

Newton method is supposed to be generated from the Taylor's series expansion of $f(x)$ about any point x_1,

$$f(x) = f(x_1) + (x - x_1)f'(x_1) + \frac{1}{2!}(x - x_1)^2 f''(x_1) + \cdots$$

Here it is clear the $f, f', f'' \ldots$ are calculated over x_1. Taking first two term of above series expansion, we have-

$$f(x) = f(x_1) + (x - x_1)f'(x_1)$$

And then equate this equation to 0 i.e. $f(x) = 0$ in order to find the root of the equation-

$$f(x_1) + (x - x_1)f'(x_1) = 0$$

Rearranging this equation, we get-

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Hence we conclude that the general formula for determining root of equation using Newton method is-

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \text{where } k \in N$$

<u>*Convergence of Newton's Method:*</u>

In Newton-Raphson method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Newton's method is an iterative method in which

$x_{k+1} = \emptyset(x_k)$ and $\emptyset(x_k) = x_k - \frac{f(x_k)}{f'(x_k)}$

Also if $x_k = x$ then the equation can be written as

$$\emptyset(x) = x - \frac{f(x)}{f'(x)}$$

The consecutive approximation x_k is supposed to converge at a precise value if $|\emptyset'(x)| < 1$

$$\Rightarrow \left|1 - \frac{[f'(x)^2 - f(x)f"(x)]}{[f'(x)^2]}\right| < 1$$

$$\Rightarrow \left|\frac{f(x)f"(x)}{[f'(x)^2]}\right| < 1$$

Hence it is concluded that Newton's Method converges to an exact value if and only if $|f(x)f"(x)| < [f'(x)^2]$

3. Results and Discussion

Exemplifying the results of finding roots of equation $f(x) = x - \cos x$ in interval [0,1] by Bisection method, Newton's method and Regula-falsi method using MATLAB in this work, the results are featured in the tabular form as well as in the graphical form below.

Table-1: ***Iteration Statsfor Bisection Method***

Iteration No.	a	$f(a)$	b	$f(b)$
0	0	-1	1	0.459698

1	0.5	-0.377583	1	0.459698
2	0.5	-0.377583	0.75	0.0183111
3	0.625	-0.185963	0.75	0.0183111
4	0.6875	-0.0853349	0.75	0.0183111
5	0.71875	-0.0338794	0.75	0.0183111
6	0.734375	-0.00787473	0.75	0.0183111
7	0.734375	-0.00787473	0.7421875	0.00519571
8	0.73828125	-0.00134515	0.7421875	0.00519571
9	0.73828125	-0.00134515	0.740234375	0.00192387
10	0.73828125	-0.00134515	0.7392578125	0.000289009
11	0.73876953125	-0.000528158	0.7392578125	0.000289009
12	0.739013671875	-0.000119597	0.7392578125	0.000289009
13	0.739013671875	-0.000119597	0.7391357421875	0.0000847007
14	0.73907470703125	-0.0000174493	0.7391357421875	0.0000847007
15	0.73907470703125	-0.0000174493	0.739105224609375	0.0000336253
16	0.73907470703125	-0.0000174493	0.7390899658203125	8.08791×10^{-6}
17	0.7390823364257813	-4.68074×10^{-6}	0.7390899658203125	8.08791×10^{-6}
18	0.7390823364257813	-4.68074×10^{-6}	0.7390861511230469	1.70358×10^{-6}
19	0.7390842437744141	-1.48858×10^{-6}	0.7390861511230469	1.70358×10^{-6}
20	0.7390842437744141	-1.48858×10^{-6}	0.7390851974487305	1.07502×10^{-7}
21	0.7390847206115723	-6.90538×10^{-7}	0.7390851974487305	1.07502×10^{-7}
22	0.7390849590301514	-2.91518×10^{-7}	0.7390851974487305	1.07502×10^{-7}
23	0.7390850782394409	-9.2008×10^{-8}	0.7390851974487305	1.07502×10^{-7}
24	0.7390850782394409	-9.2008×10^{-8}	0.7390851378440857	7.74702×10^{-9}
25	0.7390851080417633	-4.21305×10^{-8}	0.7390851378440857	7.74702×10^{-9}
26	0.7390851229429245	-1.71917×10^{-8}	0.7390851378440857	7.74702×10^{-9}
27	0.7390851303935051	-4.72236×10^{-9}	0.7390851378440857	7.74702×10^{-9}
28	0.7390851303935051	-4.72236×10^{-9}	0.7390851341187954	1.51233×10^{-9}
29	0.7390851322561502	-1.60501×10^{-9}	0.7390851341187954	1.51233×10^{-9}
30	0.7390851331874728	-4.63387×10^{-11}	0.7390851341187954	1.51233×10^{-9}
31	0.7390851331874728	-4.63387×10^{-11}	0.7390851336531341	7.32998×10^{-10}
32	0.7390851331874728	-4.63387×10^{-11}	0.7390851334203035	3.43329×10^{-10}
33	0.7390851331874728	-4.63387×10^{-11}	0.7390851333038881	1.48495×10^{-10}
34	0.7390851331874728	-4.63387×10^{-11}	0.7390851332456805	5.10784×10^{-11}
35	0.7390851331874728	-4.63387×10^{-11}	0.7390851332165767	2.36988×10^{-12}
36	0.7390851332020247	-2.19844×10^{-11}	0.7390851332165767	2.36988×10^{-12}
37	0.7390851332093007	-9.80727×10^{-12}	0.7390851332165767	2.36988×10^{-12}
38	0.7390851332129387	-3.71869×10^{-12}	0.7390851332165767	2.36988×10^{-12}
39	0.7390851332147577	-6.7446×10^{-13}	0.7390851332165767	2.36988×10^{-12}
40	0.7390851332147577	-6.7446×10^{-13}	0.7390851332156672	8.47655×10^{-13}
41	0.7390851332147577	-6.7446×10^{-13}	0.7390851332152124	8.65974×10^{-14}
42	0.7390851332149850	-2.93876×10^{-13}	0.7390851332152124	8.65974×10^{-14}
43	0.7390851332150987	-1.03584×10^{-13}	0.7390851332152124	8.65974×10^{-14}
44	0.7390851332151556	-8.54872×10^{-15}	0.7390851332152124	8.65974×10^{-14}
45	0.7390851332151556	-8.54872×10^{-15}	0.7390851332151840	3.90799×10^{-14}
46	0.7390851332151556	-8.54872×10^{-15}	0.7390851332151698	1.53211×10^{-14}
47	0.7390851332151556	-8.54872×10^{-15}	0.7390851332151627	3.44169×10^{-15}
48	0.7390851332151591	-2.55351×10^{-15}	0.7390851332151627	3.44169×10^{-15}
49	0.7390851332151591	-2.55351×10^{-15}	0.7390851332151609	4.44089×10^{-16}
50	0.7390851332151600	-1.11022×10^{-15}	0.7390851332151609	4.44089×10^{-16}
51	0.7390851332151605	-3.33067×10^{-16}	0.7390851332151609	4.44089×10^{-16}
52	0.7390851332151607	0	0.7390851332151607	0

Table-2: ***Iteration Stats for Regula-Falsi Method***

Iteration No.	a	$f(a)$	b	$f(b)$
1	0	-1	1	0.45969769413186
2	0.6850733573260452	-0.0892992764818598	1	0.45969769413186
3	0.736298997613654	-0.004660039038143	1	0.45969769413186
4	0.7389453559657134	-0.000233925665771	1	0.45969769413186
5	0.738586334571446	-0.000834703454108	1	0.45969769413186
6	0.7390601403028729	-0.000041828207814	1	0.45969769413186
7	0.7390838812390941	-2.09532×10^{-6}	1	0.45969769413186
8	0.7390850705004306	-1.0496×10^{-7}	1	0.45969769413186
9	0.7390851300736199	-5.25772×10^{-9}	1	0.45969769413186
10	0.7390851330577919	-2.63374×10^{-10}	1	0.45969769413186
11	0.7390851332072771	-1.31939×10^{-11}	1	0.45969769413186
12	0.7390851332147646	-0.6624×10^{-12}	1	0.45969769413186
13	0.7390851332151409	-0.0331×10^{-12}	1	0.45969769413186
14	0.7390851332151605	-5×10^{-16}	1	0.45969769413186
15	0.7390851332151607	0	1	0.45969769413186

Table-3: ***Iteration Stats for Newton Raphson Method taking $x_0 = 0.5$***

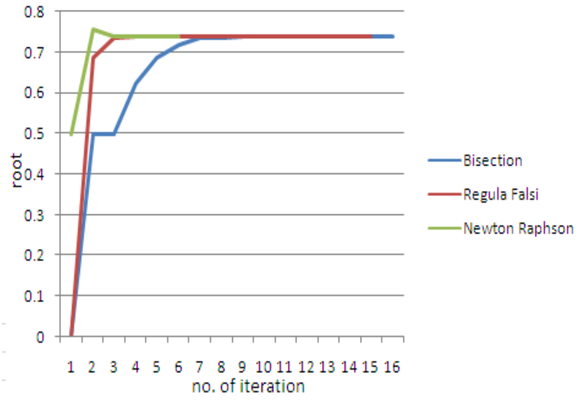
Iteration No.	x_k	x_{k+1}
1	0.5	0.7552224171056366
2	0.7552224171056366	0.7391416661498792
3	0.7391416661498792	0.7390851339208065
4	0.7390851339208065	0.7390851332151606
5	0.7390851332151606	0.7390851332151607
6	0.7390851332151607	0.7390851332151607

In table-1, the iteration stats using Bisection method by the help of MATLAB R2016a is shown. By this table we have seen that using Bisection method for the function $f(x) = x - \cos x = 0$ in interval$[0,1]$, $f(x)$ converges to 0.7390851332151607 at 52^{nd} iteration and have an error of 0.0000000.Table-2 shows that the function $f(x) = x - \cos x$ in the interval [0,1] , converges to 0.7390851332151607 and have an error 0.0000000.Table-3 shows that by taking $x_0 = 0.5$, function $f(x) = x - \cos x = 0$ converges to 0.7390851332151607 at 6^{th} iteration and have an error 0.0000000.

After comparing the results obtained by iteration data and graphical representation, we discovered that converging rates of all three methods can be written in following manner:

Newton-Raphson method > Regula-falsi method > Bisection method

By this representation and the study, we may abstract the idea that using an iterative method with higher rate of convergence we may find the root of the equation in less iteration as compared to the methods with lesser rate of convergence.



4. Conclusion

On the basis of our results, we may reach to the conclusion that Newton-Raphson method is conventionally the best and most effective methods among all the methods discussed in this paper. This conclusion is followed by the fact that Newton's method has highest rate of convergence. We also reach to a conclusion that Bisection method and Regula-falsi method are certain to converge but their converging rate is low which makes it difficult to use them to solve the system of equations.

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