Pigeonhole Principle

Theorem (Pigeonhole Principle)
If n pigeons are put into k pigeonholes, there exists at least 1 pigeonhole containing at least $\lceil \frac{n}{k} \rceil$ pigeons and at least 1 pigeonhole containing at most $\lfloor \frac{n}{k} \rfloor$ pigeons.

Proof. First, assume that every pigeonhole contains at most $\lceil \frac{n}{k} \rceil - 1$ pigeons. Since

$$\lceil x \rceil - 1 < x$$

for all real x, then there are less than

$$k\left(\left\lceil \frac{n}{k} \right\rceil - 1\right) < k\left(\frac{n}{k}\right) = n$$

pigeons. This is a contradiction. The second part can be proven similarly with

$$\lfloor x \rfloor + 1 > x$$

for all real x.

Corollary (Usual definition)
If $n+1$ pigeons are put into n pigeonholes, then there exists one hole that contains at least 2 pigeons.

Remark
In contest problems, it is usually designed so that there is exactly 1 more pigeon than the number of pigeonholes.

Example 1
51 distinct numbers are chosen from the integers between 1 and 100, inclusively. Prove that 2 of these 51 numbers are consecutive.

Proof. Let the 50 pairs of consecutive integers

$$\{1,2\}, \{3,4\}, \ldots, \{99,100\}$$

be the pigeonholes and the 51 numbers be the pigeons. Then two of the 51 numbers must be in the same pigeonholes. Therefore, there are 2 consecutive integers among the 51 chosen integers.

Example 2
10 distinct integers are chosen from 1 to 100, inclusively. Prove that there exists 2 disjoint non-empty subsets of the chosen integers such that the 2 subsets have the same sum of elements.

Proof. The maximum possible sum of the 10 integers is

$$91 + 92 + \cdots + 100 = 955$$

Therefore, let

$$1, 2, 3 \ldots, 955$$

be the pigeonholes. Next, there are

$$2^{10} - 1 = 1023$$

number of distinct nonempty subsets. These will be the pigeons. Therefore, there exists at least 1 pigeonhole with 2 pigeons. Call these two sets A and B. Next, observe that $A \backslash (A \cap B)$ and $B \backslash (A \cap B)$ are disjoint sets with the same sum. Therefore, there exists two disjoint subsets with the same set.

Example 3
Prove that if 9 distinct points are chosen in the integer lattice $\mathbb{Z}^3$, then the line segment between some two of the 9 points contains another point in $\mathbb{Z}^3$.

Proof. Reduce each coordinate of the 9 points modulo 2 to get 9 points in $\mathbb{Z}_2^3$. Since $\mathbb{Z}_2^3$ only has 8 points, then two of the 9 points, a and b, must be the same element in $\mathbb{Z}_2^3$. Thus, $a + b = 0 \in \mathbb{Z}_2^3$. This means that all three entries of $a + b \in \mathbb{Z}^3$ are even. Therefore, $\frac{1}{2}(a + b) \in \mathbb{Z}^3$.

Example 4
Prove that for every set $X = \{x_1, \ldots, x_n\}$ of n real numbers, there exists a nonempty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}$$

Proof. First, for a number x define $[x] := x - \lfloor x \rfloor$. Now, consider

$$[x_1], [x_1 + x_2], [x_1 + x_2 + x_3], \ldots, [x_1 + x_2 + \cdots + x_n]$$

If any of these are in

$$\left[0, \frac{1}{n+1}\right], \left[\frac{n}{n+1}, 1\right]$$

then we are done. If not, consider the n numbers as the pigeons and the $n - 1$ intervals

$$\left[\frac{1}{n+1}, \frac{2}{n+1}\right], \left[\frac{2}{n+1}, \frac{3}{n+1}\right], \left[\frac{3}{n+1}, \frac{4}{n+1}\right], \ldots, \left[\frac{n-1}{n+1}, \frac{n}{n+1}\right]$$

as the pigeonholes. Therefore, there exists two numbers in the same interval,

$$x_1 + \cdots + x_k, x_1 + \cdots + x_{k+m}$$

Taking their difference gives

$$x_{k+1} + \cdots + x_{k+m}$$

which lies within a distance of $\frac{1}{n+1}$ of an integer.

Example 5
Given nine points inside the unit square, prove that some three of them form a triangle whose area does not exceed $\frac{1}{8}$.

Proof. First, divide the square into four equal squares. Since there are 9 points, by Pigeonhole Principle, there exists a box with at least 3 points. It remains to show that given three points in a square, the area is at most $\frac{1}{2}$ of the area of the square.
Cut the square into two rectangles by passing a line through a vertex of the triangle. Since the area of a triangle is $\frac{1}{2}bh$ and the area of a rectangle is bh. Therefore, in each of the rectangles, the triangle has an area of at most $\frac{1}{2}bh$. Adding the two inequalities for both rectangles, the result follows.

Example 6
6 points are drawn on a plane such that no three are collinear. All edges between every pair of points are painted red or blue. Prove that there is a monochromatic triangle.

Proof. First, pick one point and call this point A. By Pigeonhole Principle, there are 3 edges painted in the same color. Without loss of generality, assume this colour is red and the three edges are AB, AC, and AD. If any of BC, BD, or CD is red, then there exists a red triangle. Otherwise, BC, BD, and CD are all blue which forms a blue triangle.

Practice Problems

1. Given 50 distinct positive integer strictly less than 100, prove that some two of them sum to 99.
2. Given 11 integers. Prove that two of them have the same unit digit.
3. There are 5 points in a square of side length 2. Prove that there exists 2 of them having a distance not more than $\sqrt{2}$.
4. 27 points are aligned so that each row has 9 points and each column has 3 points. Each point is painted red or blue. Prove that there exists a monochromatic rectangle.
5. Prove that at any party there are two people who have the same number of friends at the party (assume that all friendship are mutual).
6. Let S be a set of n integer. Prove that there is a subset of S, the sum of whose elements is a multiple of n.
7. Prove that if 101 integers are chosen from the set $\{1, 2, 3, \ldots, 200\}$ then one of the chosen integers divides another.
8. Prove that for some integer $k > 1$, 3^k ends with 0001.
9. Let n be a positive integer. Show that there is a positive multiple of n whose digits are all 0's and 1's.
10. Define $[x] := x - \lfloor x \rfloor$ for all $x \in \mathbb{R}$. Let $0 < d < 1$. Prove that in the sequence

$$\left\{\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \ldots\right\}$$

    there exists a term x such that $0 < [x] < d$.

11. Show that some pair of any 5 points in the unit square will be at most $\frac{\sqrt{2}}{2}$ units apart.
12. Show that some pair of any 8 points in the unit square will be at most $\frac{\sqrt{5}}{4}$ units apart.
13. 76 points are aligned so that each row has 19 points and each column has 4 points. Each point is painted red, blue, or yellow. Prove that there exists a monochromatic rectangle.
14. A salesman sells at least 1 car each day for 100 consecutive days selling a total of 150 cars. Show that for each value of n with $1 \leq n < 50$, there is a period of consecutive days during which he sold a total of exactly n cars.