Math 3331 - ODE's

Techniques

(1) Separable $\frac{dy}{dx} = f(x)g(y)$

(2) linear $\frac{dy}{dx} + p(x)y = q(x)$ $\quad \mu = e^{\int p(x)dx}$

(3) Bernoulli $\frac{dy}{dx} + p(x)y = q(x)y^n$ $\quad u = \frac{1}{y^{n-1}}$

(4) Riccati $\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x)$

if 1st sol^n is y_1, $\quad y = y_1 + \frac{1}{u} \rightarrow$ linear ODE

(5) Homogeneous

$$\frac{dy}{dx} = f(y/x)$$

then $\frac{y}{x} = u$ or $y = xu$

will give a separable ODE

Ex $\frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$ ← ODE is Riccati

but is also homogeneous

if $y = xu$ $\quad \frac{dy}{dx} = x\frac{du}{dx} + u$

sub

$$x\frac{du}{dx} + \cancel{u} = 1 + \cancel{u} + u^2$$

$$\frac{du}{1+u^2} = \frac{dx}{x}$$

$$\tan^{-1} u = \ln|x| + c$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + c$$

$$\frac{y}{x} = \tan(\ln|x| + c)$$

$$y = x\tan(\ln|x| + c)$$

Ex 2 $\quad \dfrac{dy}{dx} = \dfrac{2x+y}{x+2y}$

Is it of the right form.

is $\quad F(x,y) = \dfrac{2x+y}{x+2y} = f(y/x)$

to check replace

$$x \to \lambda x, \quad y \to \lambda y$$

if $\quad F(\lambda x, \lambda y) = F(x,y) \quad$ (no λ)

then yes

so $\quad \dfrac{2\lambda x + \lambda y}{\lambda x + 2\lambda y} = \dfrac{\cancel{\lambda}(2x+y)}{\cancel{\lambda}(x+2y)}$ no λ

So Yes - homogeneous

let $y = xu$

$$\frac{dy}{dx} = x\frac{du}{dx} + u$$

sub $$x\frac{du}{dx} + u = \frac{2x + xu}{x + 2xu} = \frac{\cancel{x}(2+u)}{\cancel{x}(1+2u)}$$

$$x\frac{du}{dx} = \frac{2+u}{1+2u} - u = \frac{2+u-u(1+2u)}{1+2u}$$

$$x\frac{du}{dx} = \frac{2+\cancel{u}-\cancel{u}-2u^2}{1+2u}$$

$$\Rightarrow \quad \frac{1+2u}{2(1-u^2)}\,du = \frac{dx}{x}$$

factor out -ve on left side

$$\frac{1+2u}{2(u^2-1)}\,du = -\frac{dx}{x}$$

$\leftarrow$ Part frac

$$\left(\frac{1}{4}\frac{1}{u+1} + \frac{3}{4}\frac{1}{u-1}\right)du = -\frac{dx}{x}$$

Now $\int$

$$\frac{1}{4}\ln|u+1| + \frac{3}{4}\ln|u-1| = -\ln|x| + \frac{\ln c}{4}$$

$$\ln\left|\frac{y}{x}+1\right| + 3\ln\left|\frac{y}{x}-1\right| = -4\ln|x| + \ln c$$

$$\left(\frac{y}{x}+1\right)\left(\frac{y}{x}-1\right)^3 = \frac{c}{x^4}$$

a $(y+x)(y-x)^3 = c \leftarrow$ sol^n

Suppose I.C. $y(1)=2$

sub $(2+1)(2-1)^3 = c \Rightarrow c = 3$

$$\boxed{\text{Sol}^n \quad (y+x)(y-x)^3 = 3}$$

Last Technique - Exact

We'll do Case 3 on Monday Feb 6