Toronto Math Circles: Junior
Fifth Annual Christmas Mathematics Competition
Saturday, December 15, 2018
1:00 pm - 3:00 pm

Each problem is graded on a basis of 0 to 10 points. All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit. Some partial credit may be given, but only when a contestant has shown significant and substantial progress toward a solution. Calculators are not allowed.

1. Determine the sum of all multiples of 7 or 11 less than 1000?

2. Let ABC be a right triangle with integer side lengths. Determine the minimum number of side lengths that must be even.

3. An arithmetic sequence is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3,5,7,9 is an arithmetic sequence with four terms.
   In an arithmetic sequence, the average of the first and 999^{th} term is 0. Find the 500^{th} term.

4. Let $ABCD$ and $EFGH$ be two rectangles that does not overlap. Using only a ruler, provide a construction of a line that divides the area of both rectangles in half. Remember to explain why your construction works.

5. Determine the number of positive integer triplets (a, b, c) such that $a + 2b + 3c = 100$.

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1:00 pm - 3:00 pm

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1. How many 7 digit positive integers are there such that the digits are in strictly increasing or strictly decreasing order? For example: 1234567 is one such number but 1234566 is not.

2. Determine the number of positive integer triplets (a, b, c) such that $a + 2b + 3c = 100$.

3. Find the volume of the largest cylinder that can be placed in a cone with radius 5 and height 5 such that the cylinder is placed with one of its faces against the base of the cone.

4. Let $f : [0, 1] \to \mathbb{R}$ such that

$$f(0) = f(1) = 1 \text{ and } |f(a) - f(b)| < |a - b|$$

   for all $a \neq b$ in $[0, 1]$. Determine the smallest possible value real value M such that $|f(a) - f(b)| < M$.

5. Let $S(n)$ be the sum of the digits of n. Evaluate $S(S(S(2005^{2005})))$.