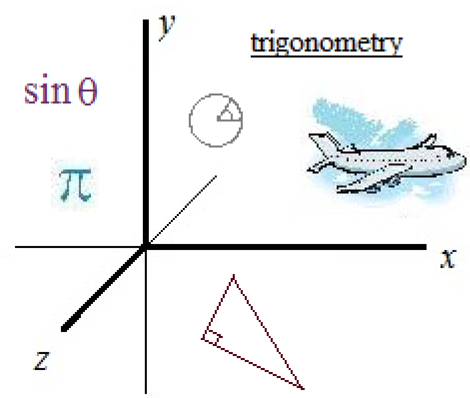
Trigonometry Identities II – Double Angles

Brief notes, formulas, examples, and practice exercises

(With solutions)



Trigonometry: *Double Angles*

What is it? Expressing trigonometric functions of angles *equal to 2x in terms of x*

For example, Sin(40) can be expressed as the double angle Sin2(20)

Why would you use them? Sometimes double angles simplify equations and make it easier to perform complex operations.

Double Angle Formulas:

$\text{Sin } 2X = 2\text{Sin}X\text{Cos}X$

$$\text{Sin}2X = \text{Sin}(X + X)$$

(Using Sum Identity)

$$= \text{Sin}X\text{Cos}X + \text{Cos}X\text{Sin}X$$

$$= 2\text{Sin}X\text{Cos}X$$

Note: $\text{Sin}2X \neq 2\text{Sin}X$

$\text{Sin}2X \neq \text{Sin}X + \text{Sin}X$

Examples:

1) $\text{Sin } 2(90) \neq 2 \text{ Sin } (90) = 2$ ✗

$\text{Sin } 2(90) = \text{Sin } (180) = 0$ ✓

$= 2 \text{ Sin } (90) \text{ Cos } (90) = 2\,(1)\,(0) = 0$ ✓

2) $\text{Sin } 2(30) \neq 2 \text{ Sin } 30 = 2 \cdot 1/2 = 1$ ✗

$\text{Sin } 2(30) = \text{Sin } 60 = \sqrt{3}/2$ ✓

or

$2 \text{ Cos}(30)\text{Sin}(30) = 2 \cdot \sqrt{3}/2 \cdot 1/2 = \sqrt{3}/2$ ✓

$\text{Cos } 2X = \text{Cos}^2 X - \text{Sin}^2 X$

$$\text{Cos}2X = \text{Cos}(X + X)$$

(Using Sum Identity)

$$= \text{Cos}X\text{Cos}X - \text{Sin}X\text{Sin}X$$

$$= \text{Cos}^2 X - \text{Sin}^2 X$$

Note: $\text{Sin}^2 X + \text{Cos}^2 X = 1$ ("Pythagorean Trig Identity")

$\text{Sin}^2 X = 1 - \text{Cos}^2 X$

$\text{Cos}^2 X = 1 - \text{Sin}^2 X$

Therefore, using substitution:

$$\text{Cos } 2X = 2\text{Cos}^2 X - 1 = 1 - 2\text{Sin}^2 X$$

3) $\text{Cos}(90) = 0$

$\text{Cos}2(45) = \text{Cos}^2(45) - \text{Sin}^2(45)$

$= \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = 0$ ✓

4) $\text{Cos}(120) = -1/2$

$\text{Cos}2(60) = \text{Cos}^2(60) - \text{Sin}^2(60)$

$= \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 = -1/2$ ✓

$\text{Cos}(120) \neq 2\text{Cos}(60) = 2(1/2) = 1$ ✗

Trigonometry: *Double Angles* (continued)

$$\text{Tan } 2X = \frac{2\text{Tan}X}{1 - \text{Tan}^2 X}$$

$$\text{Tan}2X = \text{Tan}(X + X)$$

(Using Sum Identity) $= \dfrac{\text{Tan}X + \text{Tan}X}{1 - \text{Tan}X\text{Tan}X}$

$$= \frac{2\text{Tan}X}{1 - \text{Tan}^2 X}$$

Note: $\dfrac{\text{Sin}X}{\text{Cos}X} = \text{Tan}X$ ("Quotient Trig Identity")

Therefore, it follows that $\text{Tan}2x = \dfrac{\text{Sin}2x}{\text{Cos}2x}$

5) $\text{Tan}(120) = -\sqrt{3}$

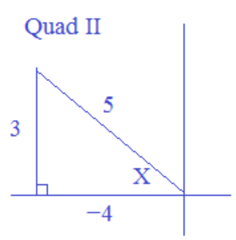
$$\text{Tan}2(60) = \frac{2\text{Tan}(60)}{1 - \text{Tan}^2(60)}$$

$$= \frac{2\sqrt{3}}{1 - (\sqrt{3})^2} = -\sqrt{3} \checkmark$$

Using Double Angle Formulas: *Practice*

1) $\text{Sin}X = \dfrac{3}{5}$ in Quadrant II

Find Sin2X, Cos2X, and Tan2X



SinX = 3/5
CosX = -4/5
TanX = -3/4

$\text{Sin}^2X = 9/25$
$\text{Cos}^2X = 16/25$
$\text{Tan}^2X = 9/16$

$$\text{Sin}2X = 2(\text{Sin}X)(\text{Cos}X) = 2\left(\frac{3}{5}\right)\left(\frac{-4}{5}\right) = \boxed{\frac{-24}{25}}$$

$$\text{Cos}2X = \text{Cos}^2X - \text{Sin}^2X = \frac{16}{25} - \frac{9}{25} = \boxed{\frac{7}{25}}$$

$$\text{Tan}2X = \frac{2\text{Tan}X}{1 - \text{Tan}^2X} = \frac{2\left(\frac{-3}{4}\right)}{1 - \left(\frac{9}{16}\right)} = \frac{\frac{-3}{2}}{\frac{7}{16}} = \boxed{-\frac{24}{7}}$$

Check Solutions:
(**Using a calculator)

Since SinX = 3/5, take the ArcSin of 3/5 (or .60)

The Reference angle X = 36.86°

Since X is in Quad II, the angle measures 180 − 36.86 = 143.14°

$\text{Sin}2(143.14) = \text{Sin}(286.28) \cong -.96$

$\text{Cos}2(143.14) = \text{Cos}(286.28) \cong .28$

$\text{Tan}2(143.14) = \text{Tan}(286.28) \cong -3.42$

Also, since $\text{Tan} = \dfrac{\text{Sin}}{\text{Cos}}$

$$\frac{\text{Sin}(2X)}{\text{Cos}(2X)} = \text{Tan}(2X)$$

$$\frac{\frac{-24}{25}}{\frac{7}{25}} = -\frac{24}{7}$$

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2) $\text{Sin}2X + \text{Sin}X = 0 \quad [0, 2\pi)$

Double Angle Identity: $2\text{Sin}X\text{Cos}X + \text{Sin}X = 0$

Factor: $\text{Sin}X(2\text{Cos}X + 1) = 0$

Solve: $\text{Sin}X = 0 \qquad 2\text{Cos}X + 1 = 0$

$\boxed{X = \pi}$ $\qquad \text{Cos}X = \dfrac{-1}{2}$

$$\boxed{X = \frac{2\pi}{3} \quad \frac{4\pi}{3}}$$

Check Solutions:
(Plug answers into original equation)

$\text{Sin}2(\pi) + \sin(\pi) = 0 + 0 = 0 \checkmark$

$\text{Sin}2\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right) = \frac{-\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = 0 \checkmark$

$\text{Sin}2\left(\frac{4\pi}{3}\right) + \sin\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{-\sqrt{3}}{2} = 0 \checkmark$

Sum and Difference Formulas

$\text{Sin}(30) = \frac{1}{2}$ $\qquad$ $\text{Sin}(60) = \text{Sin}(30 + 30)$

But, Sin(60) is NOT equal to $\frac{1}{2} + \frac{1}{2}$

$$\text{Sin}(60) = \frac{\sqrt{3}}{2}$$

$\text{Cos}(90) = 0$ $\qquad$ $\text{Cos}(30) = \text{Cos}(90 - 60)$

$\text{Cos}(60) = \frac{1}{2}$

But, Cos(30) is NOT equal to $0 - \frac{1}{2}$

$$\text{Cos}(30) = \frac{\sqrt{3}}{2}$$

Addition/Subtraction Angle Formulas (SINE)

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

Addition/Subtraction Angle Formulas (COSINE)

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

Addition/Subtraction Angle Formulas (TANGENT)

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\sin(x - y)}{\cos(x - y)} = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Verification:

$$\frac{\tan x + \tan y}{1 - \tan x \tan y} \Rightarrow \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x \sin y}{\cos x \cos y}}$$

common denominator and combine in numerator

$$\frac{\frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y}}{1 - \frac{\sin x \sin y}{\cos x \cos y}}$$

common denominator and combine in denominator

$$\frac{\frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y}}{\frac{\cos x \cos y - \sin x \sin y}{\cos x \cos y}}$$

divide the fractions

$$\frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y - \sin x \sin y}$$

addition formulas

$$\frac{\sin(x + y)}{\cos(x + y)}$$

Using the above formulas:

sin(60)

$$\sin(30 + 30) = \sin(30)\cos(30) + \cos(30)\sin(30)$$

$$= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{2}$$

$$= \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

cos(30)

$$\cos(90 - 60) = \cos(90)\cos(60) + \sin(90)\sin(60)$$

$$= 0 \cdot \frac{1}{2} + 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

Application: Find the exact value (without a calculator)

$\sin(15^\circ)$

$$\sin(45 - 30) = \sin(45)\cos(30) - \cos(45)\sin(30)$$

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$\sin(15^\circ)$ is approximately .2588

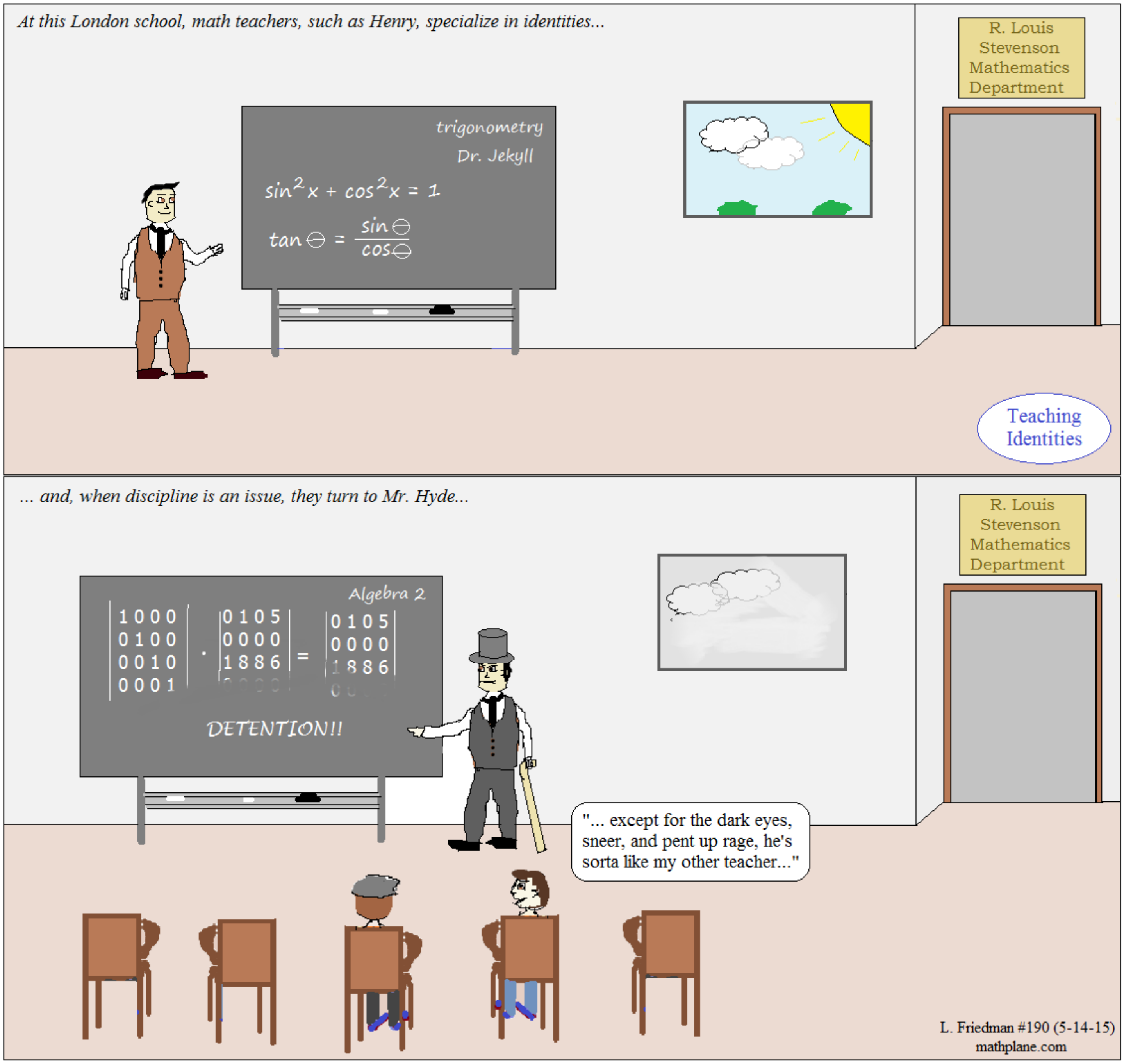
$\cos(75^\circ)$

$$\cos(30 + 45) = \cos(30)\cos(45) - \sin(30)\sin(45)$$

$$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$$

$$\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$\cos(75^\circ)$ is approximately .2588



Practice Exercise-→

Trigonometry: Double Angle Exercise

Part I: Evaluating Trig Values

1) $\sin\theta = \frac{1}{2}$ $\quad \cos\theta =$

$\tan\theta < 0$

2) $\tan X = \frac{-4}{9}$ in Quadrant II

Find the exact values of the other 5 trig functions.

3) $\cot X = 4$

$\sin X < 0$ $\quad \cos X =$

Part II: Evaluating Double Angles

1) $\sin U = \frac{-4}{5}$ $\quad \pi < U < \frac{3\pi}{2}$

Find $\sin(2U)$ and $\cos(2U)$

2) $\cot X = \frac{-7}{5}$ $\quad \frac{\pi}{2} < X < \pi$

Find $\sin(2X)$, $\cos(2X)$, and $\tan(2X)$

Trigonometry: Double Angle Exercise (continued)

III. Using Double Angle Identities

Solve the following (on the given intervals)

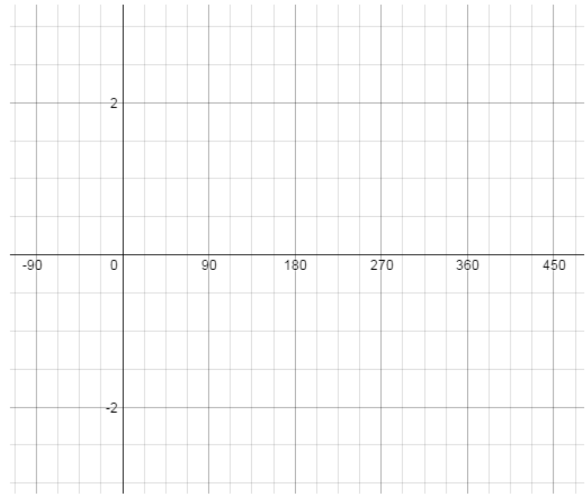
1) $\sin 2x + \sin x = 0$ $[0, 2\pi)$

2) $\cos 2x + \cos x = 0$ $[0, 2\pi)$

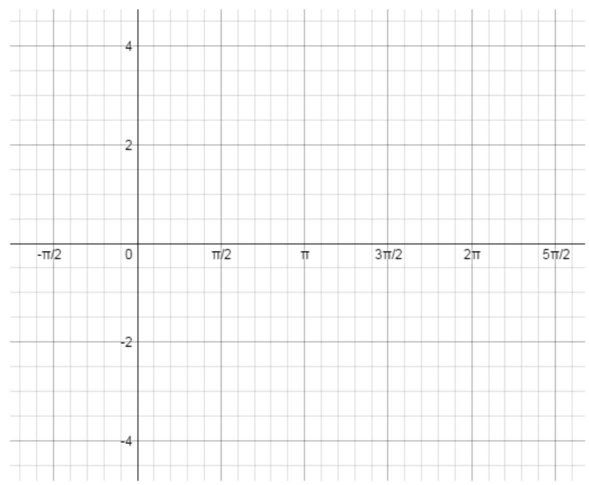
3) $4\sin\theta\cos\theta = 1$ $[0, 360^\circ)$

IV. Solve and Graph

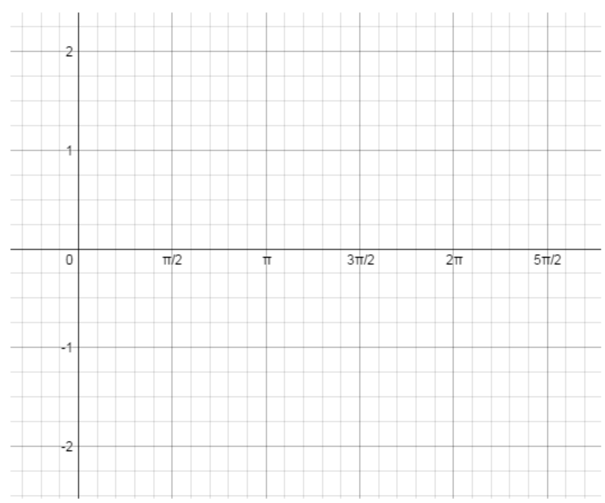
1) $\sin\theta \cos\theta = 2\cos\theta \qquad 0^{\circ} \leq \theta < 360^{\circ}$



2) $3\sin x = 1 + \cos 2x \qquad 0 \leq x < 2\pi$



3) $\sin 2x = 3\cos 2x$



Trigonometry: Double Angle Exercise

SOLUTIONS

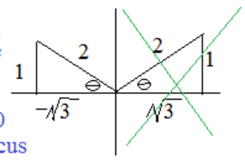
Part I: Evaluating Trig Values

1) $\sin\theta = \frac{1}{2}$

$\tan\theta < 0$

$\cos\theta = \frac{-\sqrt{3}}{2}$

$\text{Sin} = \frac{\text{opposite}}{\text{hypotenuse}}$



and, since tan < 0 in quad II, we focus on that triangle

2) $\tan X = \frac{-4}{9}$ in Quadrant II

Find the exact values of the other 5 trig functions.

using Pythagorean Theorem:

$(4)^2 + (-9)^2 = C^2$

$C = \sqrt{97}$

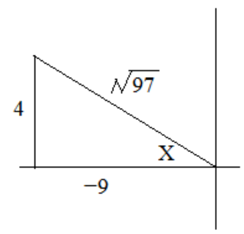
$\cot X = \frac{-9}{4}$

$\sin X = \frac{4}{\sqrt{97}}$

$\csc X = \frac{\sqrt{97}}{4}$

$\cos X = \frac{-9}{\sqrt{97}}$

$\sec X = \frac{\sqrt{97}}{-9}$



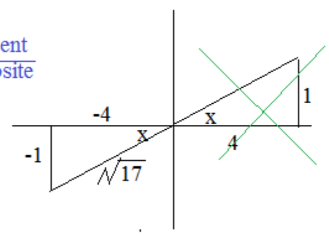
3) $\cot X = 4$

$\sin X < 0$

$\cos X = \frac{-4}{\sqrt{17}} = \frac{-4\sqrt{17}}{17}$

$\text{Cot} = \frac{\text{adjacent}}{\text{opposite}}$

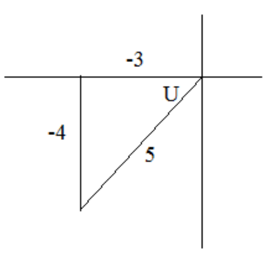
Sin < 0 in quad III



Part II: Evaluating Double Angles

1) $\sin U = \frac{-4}{5}$ $\quad \pi < U < \frac{3\pi}{2}$

Find Sin(2U) and Cos(2U)



$\sin U = \frac{-4}{5}$ $\quad \sin^2 U = \frac{16}{25}$

$\cos U = \frac{-3}{5}$ $\quad \cos^2 U = \frac{9}{25}$

$\sin(2U) = 2\sin(U)\cos(U)$

$= 2\left(\frac{-4}{5}\right)\left(\frac{-3}{5}\right)$

$= \frac{24}{25}$

$\cos(2U) = \cos^2 U - \sin^2 U$

$= \frac{9}{25} - \frac{16}{25}$

$= \frac{-7}{25}$

Note: To check solutions, use trig functions and inverse trig functions on a calculator.

$U = \text{ArcSin}(-.80) = 233.13^\circ$

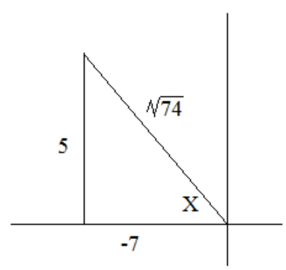
(in quad III)

$\sin(2U) = \sin 466.26^\circ = .96$ or $\frac{24}{25}$ ✓

$\cos(2U) = \cos 466.26 = -.28$ or $\frac{-7}{25}$ ✓

2) $\cot X = \frac{-7}{5}$ $\quad \frac{\pi}{2} < X < \pi$

Find Sin(2X), Cos(2X), and Tan(2X)



$\sin X = \frac{5}{\sqrt{74}}$

$\cos X = \frac{-7}{\sqrt{74}}$

$\tan X = \frac{-5}{7}$

$\sin 2X = 2\sin X\cos X = 2\left(\frac{5}{\sqrt{74}}\right)\left(\frac{-7}{\sqrt{74}}\right) = \frac{-70}{74} = \frac{-35}{37}$

$\cos 2X = \cos^2 X - \sin^2 X = \frac{49}{74} - \frac{25}{74} = \frac{24}{74} = \frac{12}{37}$

$\tan 2X = \frac{2\tan X}{1 - \tan^2 X} = \frac{2(\frac{-5}{7})}{1 - \left(\frac{-5}{7}\right)^2} = \frac{\frac{-10}{7}}{\frac{24}{49}} = \frac{-70}{24} = \frac{-35}{12}$

Note: $\frac{\sin 2x}{\cos 2x} = \tan 2x$

Trigonometry: Double Angle Exercise (continued)

SOLUTIONS

III. Using Double Angle Identities

Solve the following (on the given intervals)

1) $\text{Sin}2x + \sin x = 0 \qquad [0, 2\pi)$

$2\text{Sin}x\text{Cos}x + \sin x = 0$

factor and solve:

$\text{Sin}x\,(2\text{Cos}x + 1) = 0$

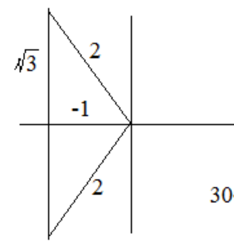
For $\text{Sin}x = 0$

$\boxed{x = 0 \text{ and } \pi}$

For $2\text{Cos}x + 1 = 0$

$\text{Cos}x = \dfrac{-1}{2}$

$\boxed{x = \dfrac{2\pi}{3} \text{ and } \dfrac{4\pi}{3}}$



30-60-90 triangles

2) $\text{Cos}2x + \cos x = 0 \qquad [0, 2\pi)$

$2\text{Cos}^2 x - 1 + \text{Cos}\,x = 0$

factor and solve:

$2\text{Cos}^2 x + \text{Cos}\,x - 1 = 0$

$(2\text{Cos}x - 1)(\text{Cos}x + 1) = 0$

For $\text{Cos}x + 1 = 0$

$\text{Cos}x = -1 \qquad \boxed{x = \pi}$

For $2\text{Cos}x - 1 = 0$

$\text{Cos}x = \dfrac{1}{2}$

$\boxed{x = \dfrac{\pi}{3} \text{ and } \dfrac{5\pi}{3}}$

3) $4\text{Sin}\theta\text{Cos}\theta = 1 \qquad [0, 360^\circ)$

$2(2\text{Sin}\theta\ \text{Cos}\,\theta) = 1$

$2(\sin 2\theta) = 1$

$\text{Sin}2\theta = \dfrac{1}{2}$

Let $U = 2\theta$

$\text{Sin}(U) = \dfrac{1}{2}$

then, $U = 30^\circ$ and 150°

AND, 390° 510° (and other coterminal angles)

Since $U = 2\theta$

$2\theta = 30^\circ, 150^\circ, 390^\circ, 510^\circ$

therefore,

$\boxed{\theta = 15^\circ, 75^\circ, 195^\circ, 255^\circ}$

IV. Solve and Graph

SOLUTIONS

1) $\sin\theta\cos\theta = 2\cos\theta \qquad 0^\circ \leq \theta < 360^\circ$

$\sin\theta\cos\theta - 2\cos\theta = 0$

$\cos\theta(\sin\theta - 2) = 0$

$\cos\theta = 0$ $\boxed{\theta = 90^\circ,\ 270^\circ}$

or

$\sin\theta - 2 = 0$

$\sin\theta = 2$ no solution

To graph, use

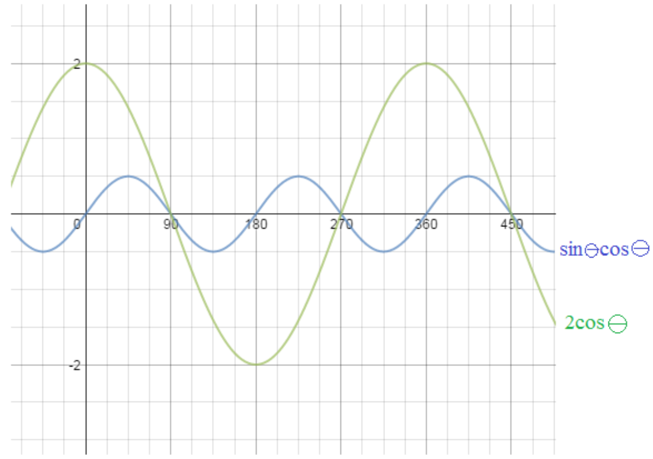
$\frac{1}{2}\sin 2\theta$ and $2\cos\theta$

the intersections are the solutions

NOTE:

$\frac{1}{2}\sin 2\theta = \frac{1}{2}(2\sin\theta\cos\theta)$

$= \sin\theta\cos\theta$



2) $3\sin x = 1 + \cos 2x \qquad 0 \leq x < 2\pi$

$3\sin x = 1 + (1 - 2\sin^2 x)$ (double angle identity)

$2\sin^2 x + 3\sin x - 2 = 0$

$(2\sin x - 1)(\sin x + 2) = 0$

$2\sin x - 1 = 0$

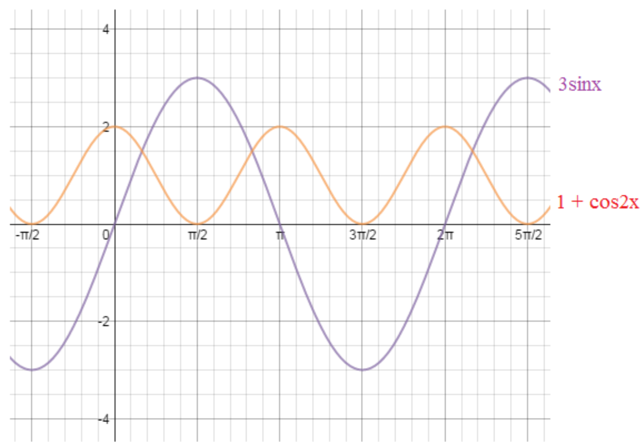
$\sin x = \frac{1}{2}$ $\boxed{x = \frac{\pi}{6}\quad \frac{5\pi}{6}}$

or

$\sin x + 2 = 0$

$\sin x = -2$ no solution

In the graph, the intersections of $3\sin x$ and $1 + \cos 2x$



3) $\sin 2x = \cos 2x \qquad 0 \leq x < 2\pi$

$\frac{\sin 2x}{\cos 2x} = 1$

$\tan 2x = 1$

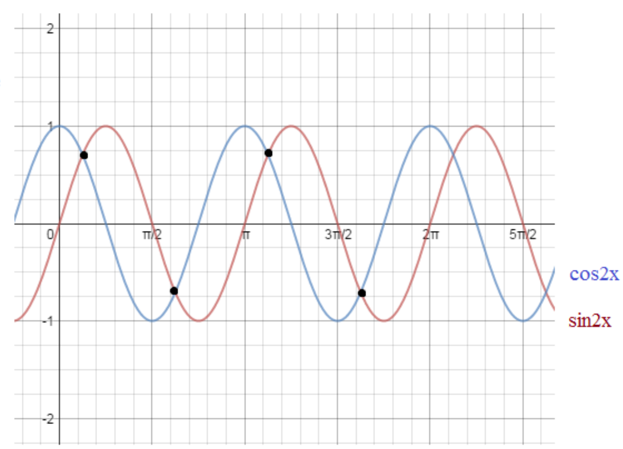
Let $A = 2x$

Then, $\tan A = 1$

$A = \frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{9\pi}{4},\ \frac{13\pi}{4},\ \ldots$

since $A = 2x$, $\boxed{x = \frac{\pi}{8},\ \frac{5\pi}{8},\ \frac{9\pi}{8},\ \frac{13\pi}{8},\ \ldots}$

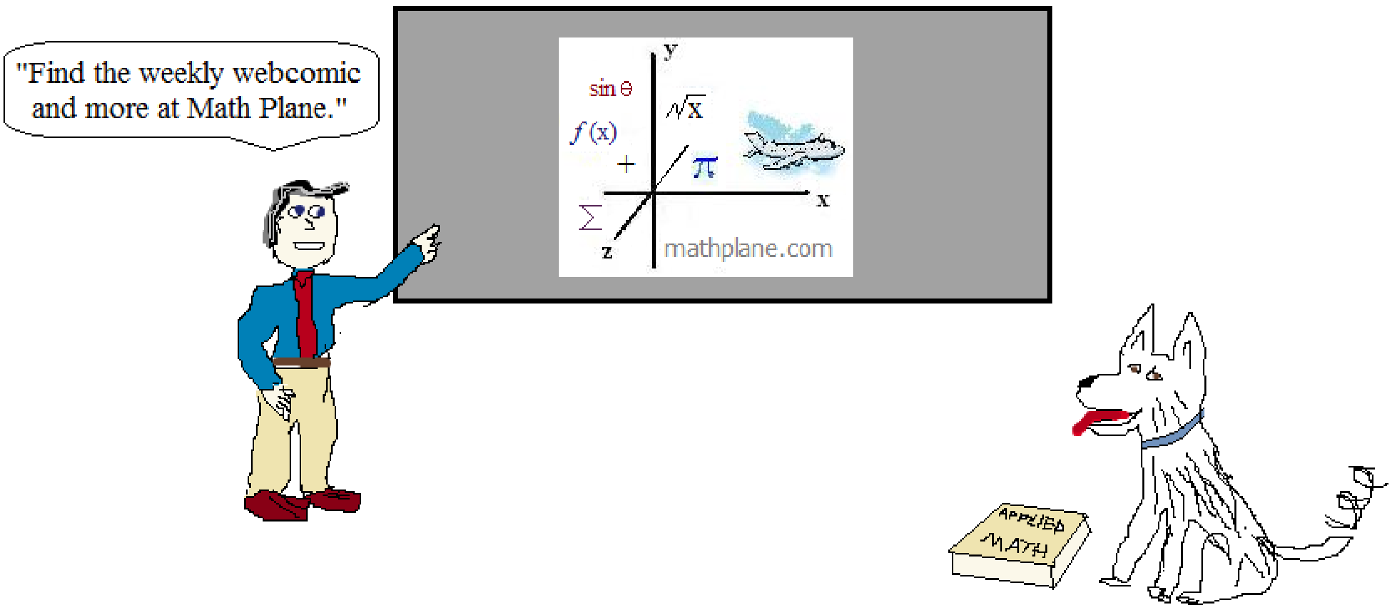
The solutions are the intersections of the two functions..



Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Good luck!



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