CAP 5993/CAP 4993
Game Theory

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Announcements

- Check calendar – there will be class on 1/31 but not on 2/2.
- HW 1: Out 1/19 due 1/26
- HW policy:
  - You can discuss general concepts with other students, but must work on the problems individually.
  - List out all resources consulted.
  - Two late days, then 50% credit, then 0%.
  - Homework due at start of class (3:30 PM). Can be emailed.

- Theorem (von Neumann): In chess, one and only one of the following must be true:
  i. White has a winning strategy
  ii. Black has a winning strategy
  iii. Each of the two players has a strategy guaranteeing at least a draw.
- Applies to ALL chess matches, not a particular match
- Theorem is significant because a priori it might have been the case that none of the alternatives was possible; one could have postulated that no player could ever have a strategy always guaranteeing a victory, or at least a draw.

Checkers is Solved (Science '07)

- The game of checkers has roughly 500 billion billion possible positions (5×10^{20}). The task of solving the game, determining the final result in a game with no mistakes made by either player, is daunting. Since 1989, almost continuously, dozens of computers have been working on solving checkers, applying state-of-the-art artificial intelligence techniques to the proving process. This paper announces that checkers is now solved: Perfect play ……

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Connect Four

Connect Four

- The solved conclusion for Connect Four is first player win. With perfect play, the first player can force a win, on or before the 41st move by starting in the middle column. The game is a theoretical draw when the first player starts in the columns adjacent to the center. For the edges of the game board, column 1 and 2 on left (or column 7 and 6 on right), the exact move-value score for first player start is loss on the 40th move, and loss on the 42nd move, respectively. In other words, by starting with the four outer columns, the first player allows the second player to force a win.

2-player limit Hold'em poker is solved (Science 2015)

Heads-up Limit Hold'em Poker is Solved

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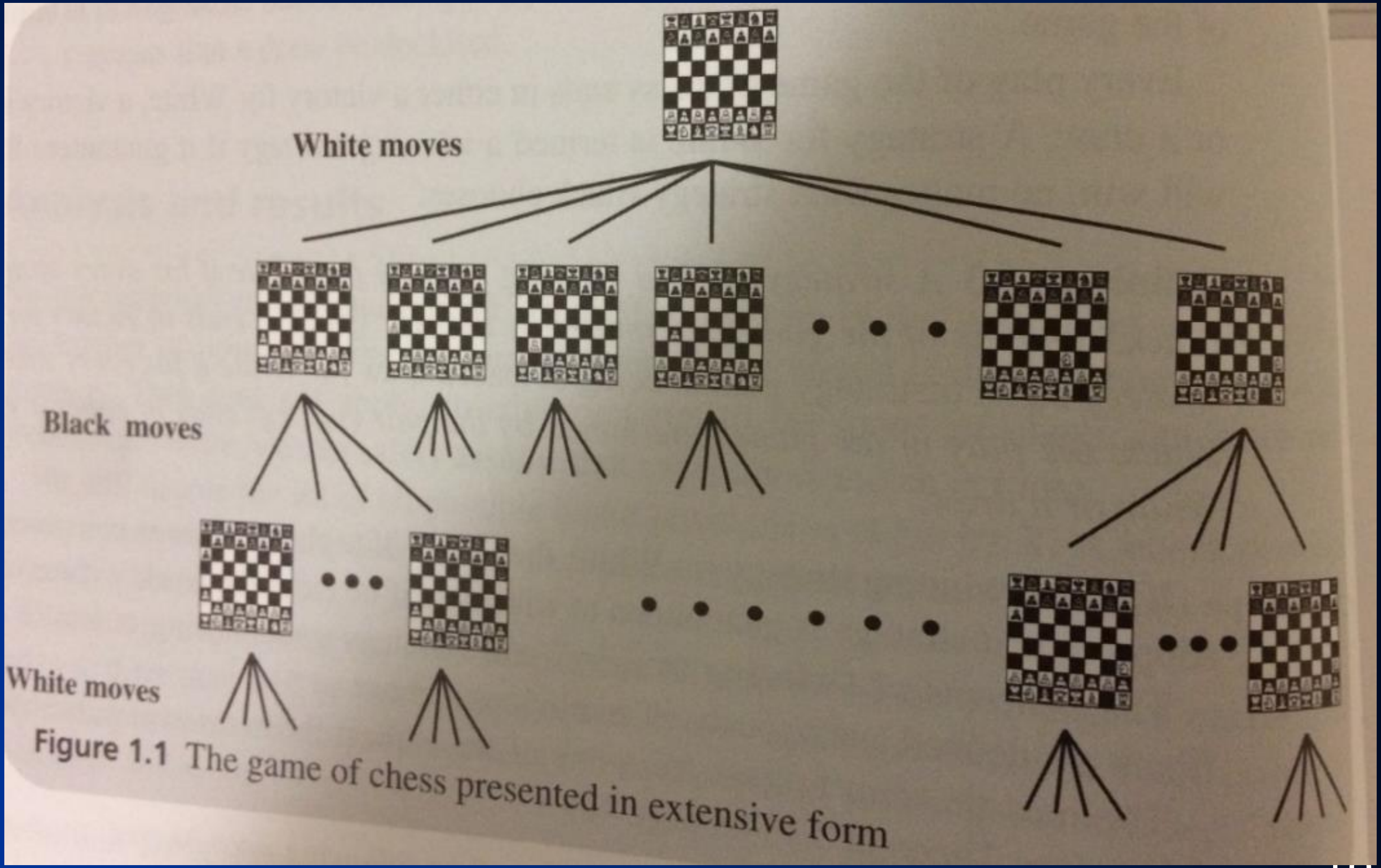
Poker is a family of games that exhibit imperfect information, where players do not have full knowledge of past events. Whereas many perfect information games have been solved (e.g., Connect Four and checkers), no nontrivial imperfect information game played competitively by humans has previously been solved. Here, we announce that heads-up limit Texas hold'em is now essentially weakly solved. Furthermore, this computation formally proves the common wisdom that the dealer in the game holds a substantial advantage. This result was enabled by a new algorithm, CFR^{+}, which is capable of solving extensive-form games orders of magnitude larger than previously possible.

Heads-up Limit Hold ‘em Poker is Solved

- Play against Cepheus here http://poker-play.srv.ualberta.ca/

Proof Sketch of Theorem

- Set of game situations can be depicted by a tree



- Denote set of vertices of game tree by H.
- The *root vertex* is the opening game situation x_0, and for each vertex x, the set of *children vertices* of x are the set of game situations that can be reached from x in one legal move.
- Every vertex that can be reached from x by a sequence of moves is called a *descendant* of x
- Every *leaf* of the tree corresponds to a terminal game situation, in which a player has won or tie

- For each vertex x, consider the *subtree* beginning at x, which is the tree whose root is x that is obtained by removing all vertices that are not descendants of x. This subtree $\Gamma(x)$ corresponds to a game that is called the *subgame beginning at x*. Denote the number of vertices in $\Gamma(x)$ by n_x. The full game is $\Gamma(x_0)$.
- F denotes set of all subgames.

- Theorem: Every game in F satisfies one and only one of the following alternatives:
    i. White has a winning strategy
    ii. Black has a winning strategy
    iii. Each of the two players has a strategy guaranteeing at least a draw.
- Proof: Induction on n_x, number of vertices in subgame $\Gamma(x)$
- For $n_x=1$, x is terminal vertex.
- Suppose $n_x>1$. Assume by induction that at all vertices satisfying $n_y< n_x$, exactly one of the alternatives holds in $\Gamma(y)$.
- …

Base case

- For n_x=1, x is terminal vertex.
- If the White King has been removed, Black has won (null strategy is winning strategy for Black).
- If the Black King has been removed, White has won (null strategy is winning strategy for White).
- If both Kings are on the board at the end of play, the game has ended in a draw, and null strategy guarantees a draw for both players.

Inductive step

- Suppose $n_x>1$. Assume by induction that at all vertices satisfying $n_y< n_x$, exactly one of the alternatives holds in $\Gamma(y)$.
- Suppose without loss of generality that White has the first move in $\Gamma(x)$. Any board position y that can be reached from x satisfies $n_y< n_x$, and so the inductive assumption is true for the corresponding subgame $\Gamma(y)$.
- Denote by C(x) the collection of vertices that can be reached from x in one of White's moves.

Inductive step

1. If there is a vertex y_0 in C(x) s.t. White has a winning strategy in $\Gamma(y_0)$, then alternative (i) is true in $\Gamma(x)$: the winning strategy for White in $\Gamma(x)$ is to choose as his first move the move leading to vertex y_0, and to follow the winning strategy in $\Gamma(y_0)$ at all subsequent moves.

Inductive step

2. If Black has a winning strategy in $\Gamma(y)$ for every vertex y in C(x), then alternative (ii) is true in $\Gamma(x)$: Black can win by ascertaining what the vertex y is after White's first move, and following his winning strategy in $\Gamma(y)$ at all subsequent moves.

Inductive step

3. Otherwise:
   - (1) does not hold, i.e., White has no winning strategy in $\Gamma(y)$ for any y in C(x). Because the IH holds for every vertex y in C(x), either Black has a winning strategy in $\Gamma(y)$, or both players have a strategy guaranteeing at least a draw in $\Gamma(y)$.
   - (2) does not hold, i.e., there is a vertex y_0 in C(x) s.t. Black does not have a winning strategy in $\Gamma(y_0)$. But because (1) does not hold, White also does not have a winning strategy in $\Gamma(y_0)$. Therefore, by the IH applied to $\Gamma(y_0)$, both players have a strategy guaranteeing at least a draw in $\Gamma(y_0)$.

Inductive step

- In case (3), White can guarantee at least a draw by choosing a move leading to vertex y_0, and from there by following the strategy that guarantees at least a draw in $\Gamma(y_0)$.
- Black can guarantee at least a draw by ascertaining what the board position y is after White's first move, and at all subsequent moves in $\Gamma(y)$, either by following a winning strategy or following a strategy that guarantees at least a draw in that subgame.

Strategic-form games

- A game in **strategic form (**or in **normal form**) is an ordered triple G = (N, (S_i) I in N, (u_i) i in N), in which:
  - N = {1,2,…,n} is a finite set of players.
  - Si is the set of strategies of player i, for every player i in N. Denote the set of all vectors of strategies by S = S_1 x S_2 x … x S_n.
  - u_i : S → R is a function associating each vector of strategies s = (s_i), i in N, with the **payoff (utility)** u_i(s) to player i, for every player i in N.

Strategic-form games

- Set of strategies available to the players are not required to be finite
- A game in which strategy set of each player is finite is called a **finite game**
- We will see examples of **infinite games**
- Important: the outcome for each player depends on the strategies chosen by ALL players, not just on his strategy alone

- Games in strategic form are sometimes called **matrix games**
- When $n = 2$, we call the games **bimatrix games**, as they are given by two matrices, one for the payoff of each player.

Chicken

- The game of chicken models two drivers, both headed for a single-lane bridge from opposite directions. The first to swerve away yields the bridge to the other. If neither player swerves, the result is a costly deadlock in the middle of the bridge, or a potentially fatal head-on collision. It is presumed that the best thing for each driver is to stay straight while the other swerves (since the other is the "chicken" while a crash is avoided). Additionally, a crash is presumed to be the worst outcome for both players. This yields a situation where each player, in attempting to secure his best outcome, risks the worst.

Chicken

	Swerve	Straight
Swerve	Tie, Tie	Lose, Win
Straight	Win, Lose	Crash, Crash

Fig. 1: A payoff matrix of Chicken

Chicken

	Swerve	Straight
Swerve	0, 0	-1, +1
Straight	+1, -1	-10, -10

Fig. 2: Chicken with numerical payoffs

Security game

- Random strategy:
  - *Increase cost/uncertainty to attackers*

Adversary

Defender

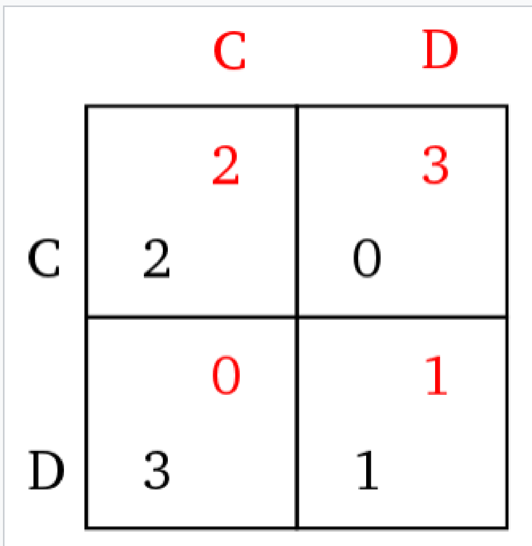
	Target #1	Target #2
Target #1	4, -3	-1, 1
Target #2	-5, 5	2, -1

Rock-paper-scissors

	rock	**paper**	**scissors**
Rock	0,0	-1, 1	1, -1
Paper	1,-1	0, 0	-1,1
Scissors	-1,1	1,-1	0,0

Prisoner’s dilemma

	Prisoner B stays silent (*cooperates*)	Prisoner B betrays (*defects*)
Prisoner A stays silent (*cooperates*)	Each serves 1 year	Prisoner A: 3 years Prisoner B: goes free
Prisoner A betrays (*defects*)	Prisoner A: goes free Prisoner B: 3 years	Each serves 2 years



A payoff matrix of the standard dilemma of cooperation and defection

Canonical PD payoff matrix

	Cooperate	Defect
Cooperate	*R*, *R*	*S*, *T*
Defect	*T*, *S*	*P*, *P*

$T > R > P > S$

Battle of the sexes

- Imagine a couple that agreed to meet this evening, but cannot recall if they will be attending the opera or a football match (and the fact that they forgot is common knowledge). The husband would prefer to go to the football game. The wife would rather go to the opera. Both would prefer to go to the same place rather than different ones. If they cannot communicate, where should they go?

	Opera	Football
Opera	3,2	0,0
Football	0,0	2,3

Battle of the Sexes 1

	Opera	Football
Opera	3,2	1,1
Football	0,0	2,3

Battle of the Sexes 2

Strategic-form game examples

- Chicken
- Security game
- Rock-paper-scissors
- Prisoner’s dilemma
- Battle of the sexes

- We saw von Neumann’s theorem in the special case of two players and three possible outcomes: victory for White, a draw, or victory for Black.
- Central question of game theory: what “will happen” in a given game?

Central question of game theory

1. An empirical, descriptive interpretation: How do players, in fact, play in a given game?
2. A normative interpretation: How "should" players play in a given game?
3. A theoretical interpretation: What can we predict will happen in a game given certain assumptions regarding "reasonable" or "rational" behavior on the part of the players?

Descriptive game theory

- Observations of the actual behavior of players, both in real-life situations and in artificial laboratory conditions where they are asked to play games and their behavior is recorded.
  - Behavioral economics, psychology

Normative interpretation

- Appropriate for a judge, legislator, or arbitrator called upon to determine the outcome of a game based on several agreed-upon principles, such as justice, efficiency, nondiscrimination, and fairness.
- Best suited for the study of cooperative games, in which binding agreements are possible, enable outcomes to be derived from "norms" or agreed-upon principles, or determined by an arbitrator who bases his decisions on those principles.

Theoretical interpretation

- After we have described a game, what can we expect to happen?
- What outcomes, or set of outcomes, will reasonably ensue, given certain assumptions regarding the behavior of the players?

- For each of the five example games we discussed:
  - How will real players act?
  - How “should” players act?
  - How would theoretically perfectly rational players act?
- Golden Balls: Split or Steal? https://www.youtube.com/watch?v=S0qjK3TWZE8

Homework for next class

- 4.8-4.15 from Maschler textbook