Math 1496 - Sample Test 2

1. Find the absolute minimum and maximum of the following on the given interval

$$\begin{array}{cc} (i) & f(x) = 1 - x^2 \quad \text{on} \quad [-1,3] \\ (ii) & f(x) = 2x^3 - 15x^2 + 24x \quad \text{on} \quad [0,3] \end{array}$$

1(i) The derivative is $f'(x) = -2x$. We set this to zero giving the critical point $x = 0$. Now we check the endpoints and the CP: $f(-1) = 0, f(0) = 1, f(3) = -8$ giving the max of 1 at $x = 0$ and min of -8 at $x = 3$

1(ii) The derivative is $f'(x) = 6x^2 - 30x + 24 = 6(x-1)(x-4)$. We set this to zero giving the critical points $x = 1, 4$ but $x = 4$ is not in the interval. Now we check the endpoints and the CP: $f(0) = 0, f(1) = 11, f(3) = -9$ giving the max of 11 at $x = 1$ and min of -9 at $x = 3$

2. State the Mean Value Theorem. Verify the Mean Value Theorem for the following

$$\begin{array}{cc} (i) & f(x) = x^3 - x \quad \text{on} \quad [0,2] \\ (ii) & f(x) = \dfrac{x}{x+2} \quad \text{on} \quad [1,10] \end{array}$$

MVT. If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there is at least one c in (a,b) where

$$\frac{f(b) - f(a)}{b - a} = f'(c) \tag{1}$$

2(i) First. $\dfrac{f(b) - f(a)}{b - a} = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{6 - 0}{2 - 0} = 3$. The derivative is $f'(x) = 3x^2 - 1$. Thus,

$$3x^2 - 1 = 3$$

giving $x = \pm\dfrac{2}{\sqrt{3}}$ and we choose c to be the positive case so $c = \dfrac{2}{\sqrt{3}}$.

2(ii) First. $\dfrac{f(b) - f(a)}{b - a} = \dfrac{f(10) - f(1)}{10 - 1} = \dfrac{\frac{10}{12} - \frac{1}{3}}{10 - 1} = \dfrac{1}{18}$. The derivative is $f'(x) = \dfrac{2}{(x+2)^2}$. Thus,

$$\frac{2}{(x+2)^2} = \frac{1}{18}$$

giving $x = 4, -8$ and we choose c to be the one in the interval so $c = 4$.

3. If $y = x^4 - 6x^2 - 8x$ calculate the following

(i) The critical numbers

(ii) When y is increasing and decreasing.

(iii) Determine whether any of the critical numbers are minimum or maximum.

(iv) When y is concave up and down and determine the points of inflection.

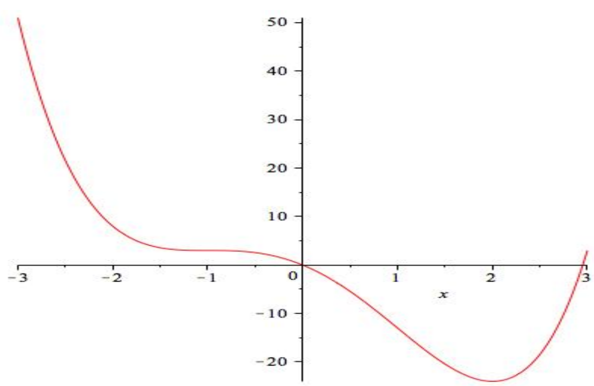
(v) Then sketch the curve.

Soln. $y' = 4x^3 - 12x - 8 = 4(x-2)(x+1)^2$. So the critical points are $x = -1, 2$. Next $y'' = 12x^2 - 12 = 12(x-1)(x+2)$ and so $y'' = 0$ when $x = -1, 1$ possible PI. Next we create the sign chart.

x		-1		1		2	
$x-2$	$-$	$-$	$-$	$-$	$-$	0	$+$
$(x+1)^2$	$+$	0	$+$	$+$	$+$	$+$	$+$
$(x-2)(x+1)^2$	$-$	0	$-$	$-$	$-$	0	$+$
slope	\	__	\	\	\	__	/
$x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$(x+1)$	$-$	0	$+$	$+$	$+$	$+$	$+$
$(x-1)(x+1)$	$+$	0	$-$	0	$+$	$+$	$+$
h/v	$\cup$	PI	$\cap$	PI	$\cup$	$\cup$	$\cup$

From here we can answer our questions.

increasing $(2, \infty)$
decreasing $(-\infty, -1)$ $(-1, 2)$
min $(2, -24)$
max - none
concave up $(-\infty, -1)$ $(1, \infty)$
concave down $(-1, 1)$
PI $(-1, 3)$ and $(1, -13)$



4. A ladder 13 feet long is resting against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 ft/sec. At rate is the tip of the ladder moving down the wall when the base of the ladder is 5 ft away from the wall?

Soln. We draw a picture of a triangle and denote the base and height of the triangle as x and y. We know $\frac{dx}{dt} = 2$ and want to know $\frac{dy}{dt}$ when $x = 5$. We first relate the variables. So $x^2 + y^2 = 169$. Then relate the rates so $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ so $\frac{dy}{dt} = -\frac{x\frac{dx}{dt}}{y}$. So we need y. Since $x^2 + y^2 = 169$, then $y = \sqrt{169 - 5^2} = 12$. Thus, $\frac{dy}{dt} = -\frac{5 \cdot 2}{12} = -\frac{5}{6} ft/sec$

5. A paper cup in the shape of an inverted cone with height 10 cm and a base of radius 3 cm, is being filled at a rate of 2 cm^3/min. Find the rate of change in the height of the water when the height of the water is 5 cm.

Soln. We draw a picture of inverted similar triangles. The larger triangle has half base of 3 and height of 10. The smaller has half base of r and height h. From the similar triangles we know that $\frac{r}{3} = \frac{h}{10}$. We know $\frac{dV}{dt} = 2$ and want to know $\frac{dh}{dt}$ when $h = 5$. We first relate the variables. So $V = \frac{1}{3}\pi r^2 h$. From the similar triangles $r = \frac{3h}{10}$ so $V = \frac{3\pi h^3}{100}$. We relate the rates so $\frac{dV}{dt} = \frac{9\pi h^2}{100}\frac{dh}{dt}$. We substitute $h = 5$ and solve for $\frac{dh}{dt}$ giving $\frac{dh}{dt} = \frac{8}{9\pi}$ cm/min.

6. A rancher has 200 feet of fencing with which to enclose two adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum?

Soln. We draw a picture of two adjacent rectangles and label width and height as x and y. The length of fence we know so

$$l = 4x + 3y = 200$$

We want to maximize area

$$A = 2xy$$

so $y = \frac{200 - 4x}{3}$ and A becomes

$$A = \frac{2x(200 - 4x)}{3}$$

Now

$$A' = \frac{400 - 16x}{3}$$

and $A' = 0$ when $x = 25$. $A'' = -\frac{16}{3} < 0$ so we have a max. So the dimensions of each pen is $25'$ by $100/3\,'$.

7. An box with a square bottom is to be built that holds 64 cubic feet. Find the dimensions of the box that will minimize the surface area of the box.

Soln. We draw a picture of a rectangular box and denote the base (square) and height as x by x and y. The volume we know so

$$V = x^2 y = 64$$

We want to maximize area

$$A = 2x^2 + 4xy = 2x^2 + \frac{4(64)}{x}$$

Now

$$A' = 4x - \frac{4(64)}{x^2}$$

We set $A' = 0$ and solve for x giving $x = 4$. Since $A'' = 4 + \frac{3(4)(64)}{x^3} > 0$ when $x = 4$ so we have a minimum. So the dimensions of the box is $4' \times 4' \times 4'$.