Calculus 3 - Vector Functions

We were introduced to vectors in Calc 2. Simply put, a vector is a directed line segment. For example, consider the points $P(1,1)$ and $Q(2,3)$. The line connecting $P \to Q$ is the vector (see figure 1). Note we have an arrow to denote it has direction.

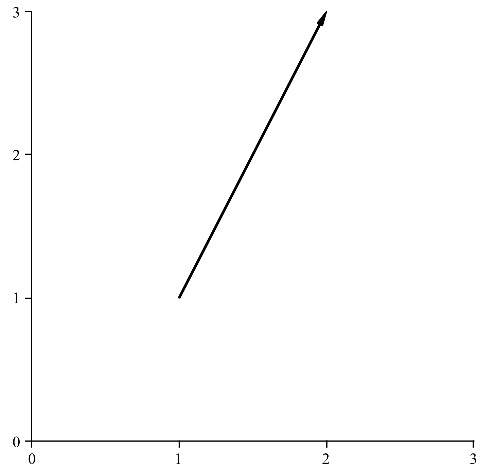

Figure 1: A vector

<u>*Symbolically*</u>

The vector in this case is $\vec{u} = \overrightarrow{PQ} =< 2-1, 3-1 >=< 1, 2 >$. The magnitude is

$$\|\vec{u}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Vector Functions

Now we extend the idea of vectors and allow the components to vary and in this case with respect to t. So we define a vector function as

$$\vec{r}(t) =< f(t), g(t) >$$

For example

$$(i) \quad \vec{r}(t) = < t, t+1 >$$
$$(ii) \quad \vec{r}(t) = < t, \tfrac{1}{4}t^2 + 1 >$$
$$(iii) \quad \vec{r}(t) = < 2\cos(t), 2\sin(t) >$$

As we did with functions, we can create a table of values. So in the first case

t	$\vec{r}(t) = < t, t+1 >$
-2	$< -2, -1 >$
-1	$< -1, 0 >$
0	$< 0, 1 >$
1	$< 1, 2 >$
2	$< 2, 3 >$

and then we draw each one

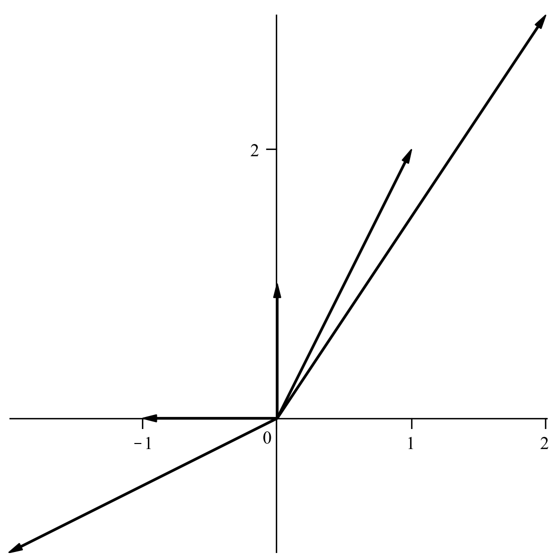

Figure 2: Vector Function $\vec{r}(t) = < t, t+1 >$

Similarly, with (ii) and (iii) we see

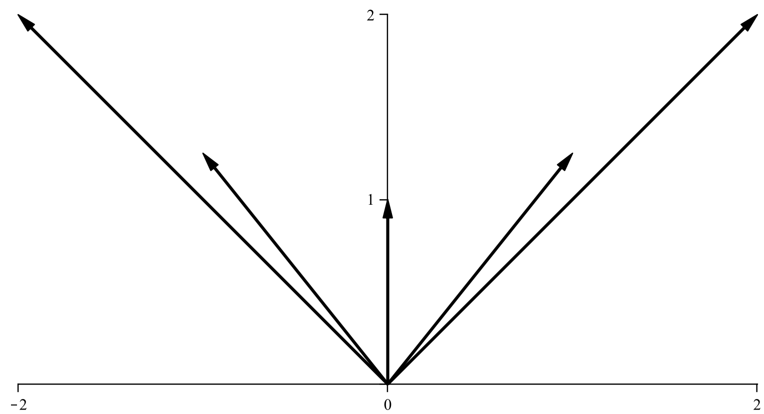

Figure 3: Vector Function $\vec{r}(t) = < t, \frac{1}{4}t^2 + 1 >$

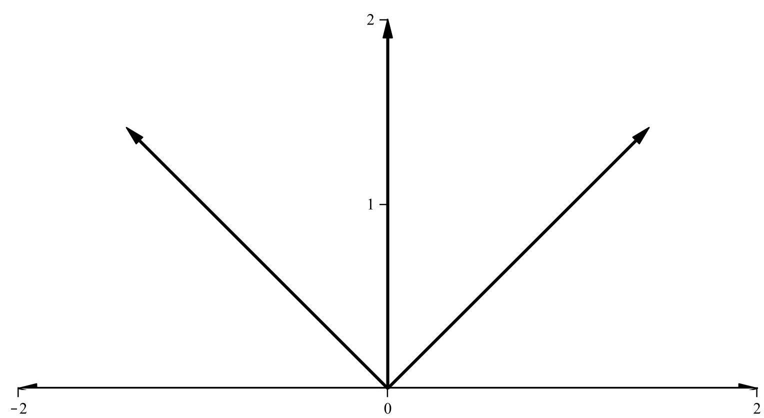

Figure 4: Vector Function $\vec{r}(t) = < 2\cos(t), 2\sin(t) >$

As we sketch the vector function, we see a curve is traced out. In example (i) we see that the curve is givien by

$$x = t, \quad y = t + 1 \tag{1}$$

and eliminating t gives

$$y = x + 1 \tag{2}$$

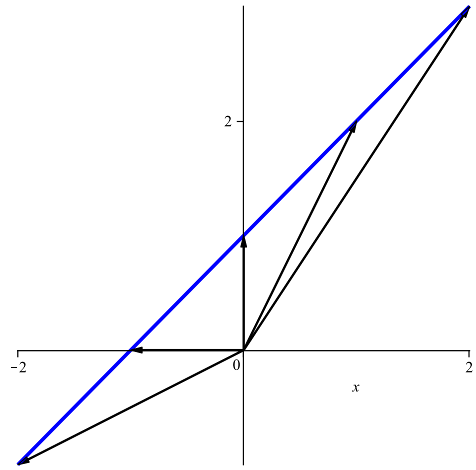

Figure 5: Vector Function(i)

Similarly in (ii)

$$x = t, \quad y = \frac{1}{4}t^2 + 1, \quad \Rightarrow \quad y = \frac{1}{4}x^2 + 1 \tag{3}$$

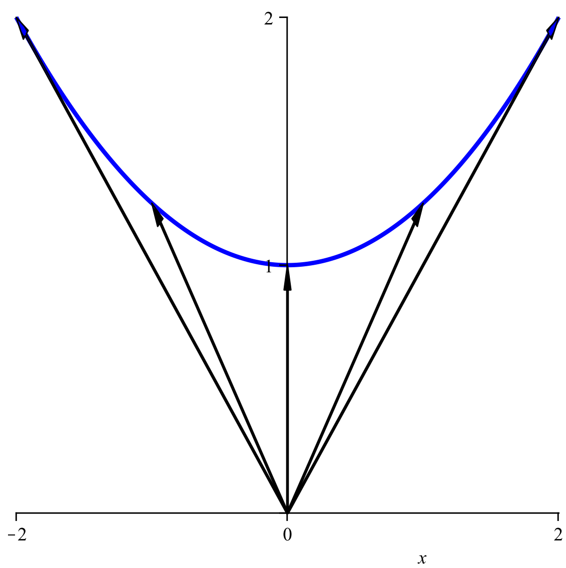

Figure 6: Vector Function (ii)

and (iii)

$$x = 2\cos(t), \quad y = 2\sin(t), \quad \Rightarrow \quad x^2 + y^2 = 4 \tag{4}$$

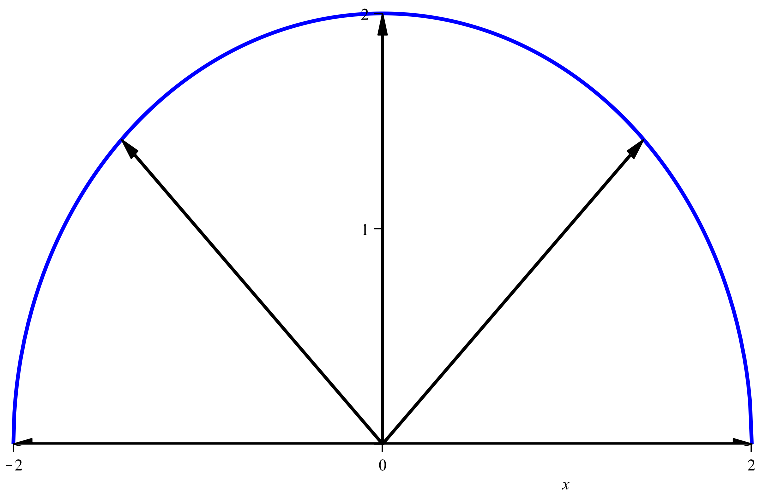

Figure 7: Vector Function (iii)

What's important to recognize is that as the vector varies, the space curve is drawn with direction.

In general, if the vector functions is

$$\vec{r} = < f(t), g(t) > \tag{5}$$

the space curve is

$$x = f(t), \quad y = g(t) \tag{6}$$

3D Vector Functions

This easily extends to $3D$. So we have

$$\vec{r} = < f(t), g(t), h(t) > \tag{7}$$

and the space curve

$$x = f(t), \quad y = g(t), \quad z = h(t) \tag{8}$$

For example consider

$$\vec{r} = < \cos(t), \sin(t), t > \tag{9}$$

The space curve here is

$$x = \cos(t), \quad y = \sin(t), \quad z = t. \tag{10}$$

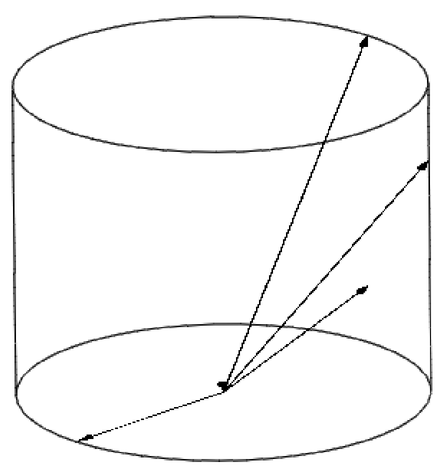

Figure 8: 3D Vector Function