VANDERBILT UNIVERSITY School of Engineering

Discrete Structures

CS 2212

(Fall 2020)

2 – Logic

Reminder and Recap …

Reminder:

- **ZyBook Assig. 1A** and **1B** due **Sep. 06** (11:59 PM)

Recap:

We are trying to develop necessary tools for logical reasoning.

- **Propositions** are statements with a definite truth value.

  (building blocks of our logical reasoning framework.)

- We can **combine** propositions using **logical operators** to get compound propositions.
- The truth value of compound propositions depend on the definition of logical operators.

1.1 Propositions and Logical Operations

Conjunction:

p and **q** are propositions

conjunction of **p** and **q** is a new **proposition**, whose truth value is

true when **both p** and **q** are *true*, and

false otherwise.

Written as:

$p \wedge q$

Read as:

p and **q**

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Truth table

p: 31 is an even number (F)

q: 31 is a prime number (T)

$p \wedge q = F$

1.1 Propositions and Logical Operations

Disjunction:

p and **q** are propositions

disjunction of **p** and **q** is a new **proposition**, whose truth value is

false when **both p** and **q** are *false*, and

true otherwise.

Written as:

$$p \lor q$$

Read as:

p or **q**

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Truth table

p: 31 is an even number (F)

q: 31 is a prime number (T)

$$p \lor q = T$$

1.1 Propositions and Logical Operations

Exclusive-or:

p and **q** are propositions

exclusive-or of **p** and **q** is a new **proposition**, whose truth value is

- ***true*** when **exactly one** of the propositions **p** and **q** is *true*
- ***false*** otherwise.

Written as:

$$\mathbf{p} \oplus \mathbf{q}$$

Read as:

p xor **q**

p	q	p ⊕ q
T	T	F
T	F	T
F	T	T
F	F	F

Truth table

p: 31 is an odd number (T)

q: 31 is a prime number (T)

$$\mathbf{p} \oplus \mathbf{q} = F$$

1.1 Propositions and Logical Operations

Let p and q be propositions, then under what conditions

1) $p \oplus q \neq p \vee q$

2) $p \vee q = p \wedge q$

3) $p \oplus q = p \wedge q$

Solution:

1. p = T, q = T
2. p = T, q = T
3. p = F, q = F

1.1 Propositions and Logical Operations

Negation:

negation of proposition **p** is a new **proposition**, whose truth value is the *opposite* of the truth value of p

false when **p** is *true*, and

true when **p** is *false*.

Written as:

¬ p

Read as:

not **p**

p	q
T	F
F	T

Truth table

p: 31 is an even number (F)

¬ p : 31 is *not* an even number (T)

1.2 Evaluating Compound Propositions

Order of operations is important.

Example: **p** = True, **q** = False,

r = ¬ p ∧ q ??

- If ¬ is first, then **r** = False
- If ∧ is first, then **r** = True

Order of operations (in the absence of parentheses):

Operator	Order
¬	1
∧	2
∨	3

1.2 Evaluating Compound Propositions

$$\mathbf{s = (p \lor q) \land \neg (p \land q)}$$

p	**q**	**p ∨ q**	**¬ (p ∧ q)**	**s**
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	T	F

1. Evaluate **p ∨ q**
2. Evaluate **p ∧ q**
3. Evaluate **¬ (p ∧ q)**
4. Evaluate the **or** of step 1 and step 3.

- Note that **s = p ⊕ q**
- **p ⊕ q** is **logically equivalent** to **(p ∨ q) ∧ ¬ (p ∧ q).**

(same truth tables)

1.2 Evaluating Compound Propositions

- **Truth table** supplies all possible truth values of a compound proposition for various truth values of its constituent proposition.
- If there are n variables, how many rows are in the truth table? 2^n

Example: $(p \land q) \lor \neg r$

How many rows in the truth table?

n = 3 variables

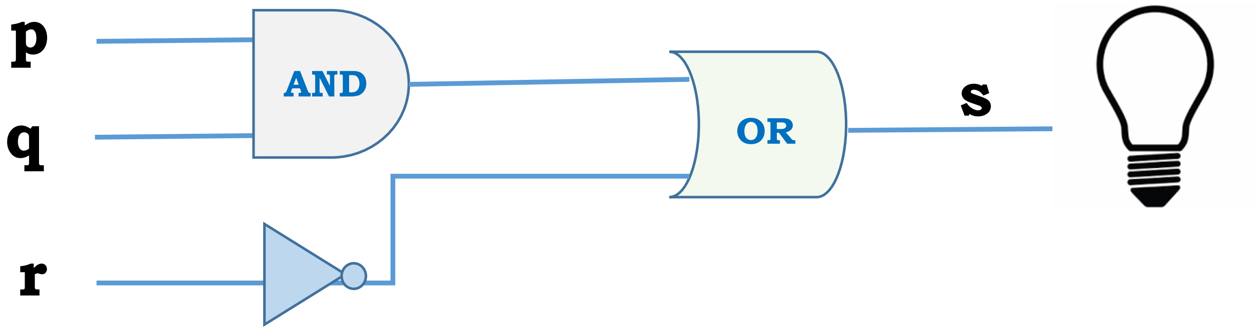
8 rows.

- Compound statements also represent digital logic circuits.

1.2 Evaluating Compound Propositions

$$s = (p \land q) \lor \neg r$$

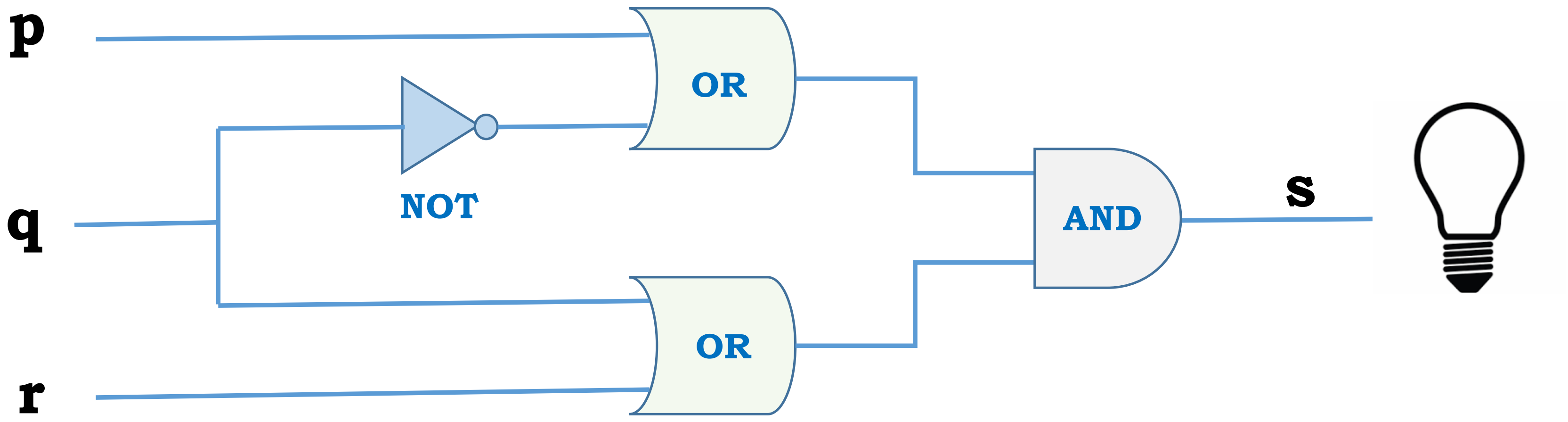
True = 1
False = 0



Example

True = **1**

False = **0**



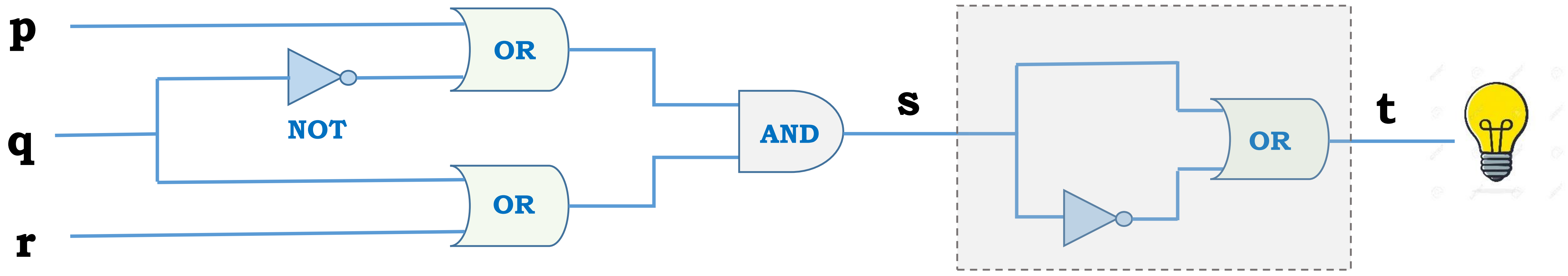
- For what values of **p**, **q**, and **r**, the bulb lights up (**s** = 1)?

$$\mathbf{s} = (\neg\, \mathbf{q} \lor \mathbf{p}) \land (\mathbf{q} \lor \mathbf{r})$$

- A solution is: **p** = **1**, **r** = **1**, **q** = **0**

Example

What can we do to ensure that bulb **always** lights up (**s = 1** irrespective of **p, q, r**) ?



$t = (\neg s \vee s)$

Tautology

s	t
1	1
0	1

1.3 Conditional Statement

Conditional Proposition:

Hypothesis → Conclusion
$p \to q$
If p, then q

Examples:

- If 29 is prime, then it is an odd number (T)
- If 29 is prime, then it is an even number (F)
- If sun rises from west, then it will rain every day (T)

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

1.3 Conditional Statement

Conditional Proposition:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

If the *conclusion* is true regardless of the *hypothesis* part, the conditional statement is **trivially true.**

p: "whatever"
q: $3 < 4$ (True)
$p \rightarrow q$: If (whatever), then $(3 < 4)$ (True)

1.3 Conditional Statement

Conditional Proposition:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

If the *conclusion* is true regardless of the *hypothesis* part, the conditional statement is **trivially true.**

p: "whatever"
q: $3 < 4$ (True)
p → q: If (whatever), then $(3 < 4)$ (True)

If the *hypothesis* is false, then the conditional statement is **vacuously true** regardless of the *conclusion* part.

p: $0 = 1$ (False)
q: "whatever"
p → q: If $(0=1)$, then "whatever". (True)

1.3 Conditional Statement

Is $p \to q$ same as (equivalent to) $q \to p$?

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

1.3 Conditional Statement

Conditional Proposition:

Is the truth value of

$$(p \vee q) \rightarrow r$$

always same as the truth value of

$$(p \rightarrow r) \wedge (q \rightarrow r) \ ?$$

Yes.

1.3 Conditional Statement

If input to the code is correct, then the output is correct

$$p \rightarrow q$$

Equivalent statements ($p \rightarrow q$)	
p implies q	Correct input implies correct output
p is sufficient for q	Correct input is sufficient for correct output
q is necessary for p	Correct output is necessary with the correct input.
p only if q	Input to the code is correct only if the output is correct

1.3 Conditional Statement

Conditional Proposition:

Hypothesis → Conclusion
$p \to q$
If p, then q

When the conditional statement ($p \to q$) is true, we say:

p is a **sufficient condition** for q

q is a **necessary condition** for p

p	q	$p \to q$
T	T	T
~~T~~	~~F~~	~~F~~
F	T	T
F	F	T

Necessary and Sufficient Conditions

Sufficient condition:

If sufficient condition is **true** then it is **guaranteed** that the conclusion holds.

> If **it rains**, then **schools are closed**.

(sufficient condition)

If it is not raining, what can we say about schools? Are they open or closed?

- We can't say anything.

Why? (see next slide)

Necessary and Sufficient Conditions

Sufficient condition:

So, given that the conditional proposition **p → q** is true (that is, we only need to consider the three rows of the truth table), following statement holds.

p	q	p → q
T	T	T
~~T~~	~~F~~	~~F~~
F	T	T
F	F	T

If sufficient condition (hypothesis) is **not true**, we **can't say** anything about the conclusion.

See that hypothesis is false in both 3rd and 4th rows. Also, **p → q** is true in both 3rd and 4th rows. However,

- conclusion is true in the 3rd row
- conclusion is false in the 4th row,

So, we can't say anything about the conclusion here.

p (hypothesis)	q (conclusion)	p → q
T	T	T
~~T~~	~~F~~	~~F~~
F	T	T
F	F	T

Necessary and Sufficient Conditions

Necessary condition:

If **it rains**, then **schools are closed**.

(necessary)

Basically, we are saying, schools are *necessarily* closed if it rains.

If schools are not closed, what can we say about rain?

- We know for a fact that it is not raining.

However, if schools are closed, what can we say about rain?

- We can't say anything.

Necessary and Sufficient Conditions

Necessary condition:

So, given that the conditional proposition **p → q** is true (that is, we only need to consider the three rows of the truth table), following statement holds.

p	q	p → q
T	T	T
~~T~~	~~F~~	~~F~~
F	T	T
F	F	T

When necessary condition (conclusion) is **not true,** it is **guaranteed** that the **hypothesis does not hold**.

See the 4th row,

- Conclusion q is false,
- Conditional statement **p → q** is true,

Observe that 4th row is the only row satisfying the above two conditions. Also the value of hypothesis is false there. So, we know for a fact that hypothesis is not true (false).

p (hypothesis)	q (conclusion)	p → q
T	T	T
~~T~~	~~F~~	~~F~~
F	T	T
F	F	T

Necessary and Sufficient Conditions

Necessary condition:

Similarly, given that the **p → q** is true, the following statement also holds.

However, if necessary condition (conclusion) is **true**, we **can't say** anything about the hypothesis.

See that conclusion is true in both 1st and 3rd rows. Also, **p → q** is true in both 1st and 3rd rows. However,

- hypothesis is true in the 1st row
- hypothesis is false in the 3rd row,

So, we can't say anything about the hypothesis.

hypothesis	conclusion	
p	**q**	**p → q**
T	T	T
~~T~~	~~F~~	~~F~~
F	T	T
F	F	T

1.3 Conditional Statement

Bi-conditional Proposition:

$$p \leftrightarrow q$$

p if and only if q

Example: The computer code works accurately if and only if all subroutines are correct.

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$$

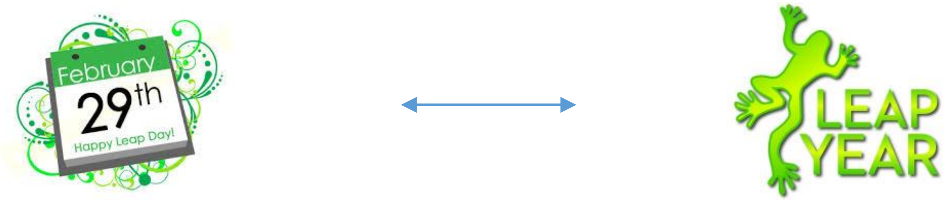
It simply means,

p and q are equivalent.

1.3 Conditional Statement

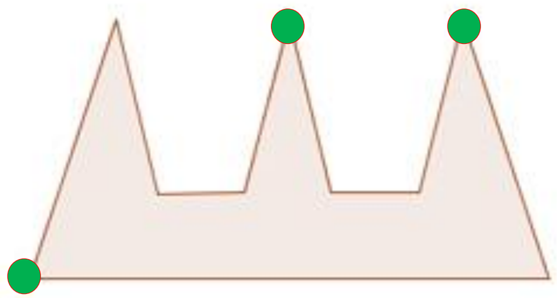
Biconditional Proposition:

Example: February has 29 days *if and only if* it's a leap year.



Example:

Three cameras are *necessary and sufficient* to completely monitor the below area.



(Art Gallery Problem)

1.4 Logical Equivalence

Propositions are **logically equivalent** means they have the **same truth values.**

- Is it true that **p** → **q** is equivalent to **¬q** → **¬p** ?

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

p	q	¬q → ¬p
T	T	T
T	F	F
F	T	T
F	F	T

- Is it true that **converse propositions** is equivalent to **inverse proposition**?

1.4 Logical Equivalence

- Show that

$(p \to q)$ is equivalent to $(\neg q \to \neg p)$?

p	q	$(p \to q)$	$\neg p$	$\neg q$	$(\neg q \to \neg p)$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Observe that 1st and 3rd columns are exactly same, hence

$$(p \to q) \quad \leftrightarrow \quad (\neg q \to \neg p)$$

1.4 Logical Equivalence

- Show that

$\neg(\mathbf{p} \rightarrow \mathbf{q})$ is equivalent to $\mathbf{p} \land \neg\mathbf{q}$?

p	q	¬(p → q)	p ∧ ¬q
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

1.4 Logical Equivalence

Tautology: If the proposition is **always true**, regardless of the truth value of individual propositions.

p	¬p	p ∨ ¬p
1	0	1
0	1	1

Contradiction: If the proposition is **always false**, regardless of the truth value of individual propositions.

p	¬p	p ∧ ¬p
1	0	0
0	1	0

Logical Equivalence

Lets construct truth table for the following:

$$(p \land q) \rightarrow p$$

p	q	p ∧ q	If (p ∧ q) , then p
T	?	?	T
F	?	F	T

Tautology

Logical Equivalence

Example: If p is a proposition, **t** is a tautology and **c** is a contradiction, then

1. **$p \wedge t$ = ?**
2. **$p \wedge c$ = ?**

Solution:

1) **$p \wedge t = p \wedge$ True = p (truth value of p)**

2) **$p \wedge c$ = False**