ONTARIO MATH CIRCLES
ANNUAL ARML TEAM SELECTION TEST
2018

Instructions

1. Do not open this competition package until you are told to do so.
2. This competition contains 40 problems to be answered in 120 minutes. This is open to all high school and middle school students.
3. Print your name clearly on your answer sheet. Phone number and e-mail addresses are requested to contact top finishers about being on the Ontario Math Circles ARML team. If you do not wish to be consider for this team, you may leave those blank. Please print these, especially e-mail address, legibly.
4. Print clearly and legibly on the answer package. You must leave all answers in exact form. For example, π would be correct while 3.1415 would not. You must simplify your answers as much as possible. For example, $\frac{6}{4}$ must be simplified to $\frac{3}{2}$ and all denominators must be rationalized. Perfect squares must be removed from radicals. For example, $\sqrt{99}$ must be written as $3\sqrt{11}$. Trig functions of standard arguments must be evaluated. Frequently, several equivalent expressions will be considered correct. For example, $\frac{3}{2}$, $1\frac{1}{2}$, and 1.5 will all be considered correct.
5. Each correct answer is worth 1 point. A blank and an incorrect answer are both worth 0 points. There are no fractional points.
6. Only pencils, erasers, and pens are allowed. **Electronic devices and calculators are not allowed.** Everything else must be approved by the proctor.
7. You may keep your question sheet and scratch work.
8. The full results of this competition will be posted on the Toronto Math Circles' website. It will only display your name, school, grade, and score.
9. The prize for top scorers will be a numerical amount to be deducted from one of their future ARML trip fees. This amount is not transferable nor redeemable for cash.
10. ***Do not discuss the problems or solutions from this contest for the next 7 days.***
11. GOOD LUCK!

1. Determine the unit's digit of $222 \times 3333 - 7777777^2$.

2. Determine the solutions to the equation $x^2 - 2x + 6 = 0$.

3. Determine the area of the triangle bounded by the y-axis and the following two lines:

$$y = -x + 1$$
$$y = 2x - 2$$

4. Thinula can make a contest in 40 minutes. Bill can make a contest in 30 minutes. Eddy hires both of them to make one contest, in how many hours does it take for the job to be done.

5. In terms of area, a regular heptagon with side length 1 is compared to a regular octagon with side length 1. Determine the area of the larger one.

6. Let $f(x) = x^4 - x^3 + ax + b$ with $f(1) = 4$ and $f(2) = 6$. Determine the ordered pair (a, b)?

7. Determine the smallest positive integer which is divisible by all one digit positive integers.

8. Let P be a point inside a convex quadrilateral $ABCD$. Determine the average of the following four angles: $\angle APB, \angle BPC, \angle CPD, \angle DPA$.

9. Determine the number of positive perfect square divisors of 10!.

10. The permutations of ELDYD are listed in lexigraphical order. Determine the 26^{th} permutation in this list.

11. Consider all lattice points on the Cartesian plane that are of distance 5 from the origin. Determine the perimeter of this convex polygon.

12. How many three digit numbers contain the digit "9" at least once?

13. Let a and b be the solutions to $x^2 - 2x + 6 = 0$. Determine the value of $a^6 + b^6$.

14. Determine all positive integer solutions (x, y, z) to the system of equations

$$\begin{cases} xy + yz = 63 \\ xz + yz = 23 \end{cases}$$

15. Determine the sum of all positive real numbers x for which $\log_4 x - \log_x 16 = \frac{7}{6} - \log_x 8$.

16. Determine the number of real solutions to $\log_4 x = 2 \sin x$.

17. The distance between the centers of two circles is 15. One has radius 4 and the other has radius 5. What is the length of their common internal tangent?

18. Let $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Compute the value of $\sum\limits_{n=1}^{\infty} \dfrac{F_n}{F_{n-1}F_{n+1}}$.

19. Let $f(x) = \frac{x}{\sqrt{1+x^2}}$ and define $f^{(n)}(x) = \underbrace{f(f(f \cdots f}_{n}(x)))$. Determine $f^{(99)}(1)$.

20. If the sum of all positive divisors of 30^{120} which are multiples of 30^{118} is $30^{118}N$, compute the value of N.

21. Let a and b be positive integers such that $a > b + 1 > 2$. If $\binom{13}{5} + \binom{13}{6} = \binom{a}{b}$, compute the maximum possible value of $a + b$.

22. If $\sin x + \cos x = \frac{1}{5}$ and $0 \leq x < \pi$, determine $\tan x$.

23. A fair die is rolled repeatedly. Given that 6 is obtained for the first time on the second roll, compute the expected number of rolls to obtain a 5 for the first time.

24. A point P is inside unit square $ABCD$. Let a, b, c, d be the area of $\triangle APD$, $\triangle DCP$, $\triangle BCP$, and $\triangle ABP$, respectively. Determine the length of the curve of points of P such that

$$\begin{cases} ax + by = 2017 \\ cx + dy = 2018 \end{cases}$$

does not have a distinct solution.

25. Let $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Determine the value of $\sum_{n=0}^{\infty} \frac{F_n}{4^n}$.

26. Evaluate $\sqrt[3]{20 + 14\sqrt{2}} + \sqrt[3]{20 - 14\sqrt{2}}$.

27. In $\triangle ABC$, $\angle A = 90^\circ$, $AC = 1$, $AB = 5$. Point D lies on ray $\vec{AC}$ such that $\angle DBC = 4\angle CBA$. Compute AD.

28. Let x be the smallest real value such that $\lfloor x^2 \rfloor - \lfloor x \rfloor^2 = 2017$. Determine $\lfloor x \rfloor$.

29. Let $f(n) = n(n+1)(n+2)(n+3)(n+4)$. If $T = \displaystyle\sum_{n=1}^{2016} f(n)$, and S is the sum of all factors of T, compute the remainder when S is divided by 50.

30. Determine all ordered triples (x, y, z) of nonnegative real numbers such that

$$\begin{cases} xy + yz + xz = 12 \\ xyz = x + y + z + 2 \end{cases}$$

31. Let $f(x) = x^3 - 2018x + 1$ and $g(x) = \frac{x-1}{x^2}$. Compute the number of, not necessarily distinct, real solutions to $f(g(x)) = 0$.

32. Let n be a natural number and let $S(n)$ denote the sum of the digits of n in its base 10 representation. Determine all natural numbers n such that $n^3 = 8S(n)^3 + 6nS(n) + 1$.

33. A portion of Thinula's test paper is cut off and he can only see the first three terms of

$$f(x) = x^{10} - 10x^9 + 45x^8$$

Given that all the roots of the polynomial are real, determine the product of the roots.

34. In $\triangle ABC$, AD is an angle bisector, D is on BC. A circle centred at O is inscribed in $\triangle ABD$. Let E be on AB such that OE is perpendicular to AB. If $BE = 2$, $BD = 3$, $AE = 4$, compute AC.

35. Let $f(x)$ be a degree 8 polynomial such that $f(k) = 2^k$ for $k = 0, 1, 2, \ldots, 8$, compute the value of $f(9)$.

36. Thinula has some coins $C_1, C_2, \ldots, C_n$. Each coin is biased so that the probability of getting heads on coin C_k is $\frac{1}{k+2}$ for all $1 \leq k \leq n$. When Thinula tosses all the coins, the probability of getting an odd number of heads is $\frac{2015}{4032}$. How many coins does Thinula have?

37. Define the sequence $a_1, a_2, \ldots$ to be a sequence such that $a_1 = 1$ and for $n \geq 1$,

$$16a_{n+1} = 1 + 4a_n + \sqrt{1 + 24a_n}$$

Evaluate $\sum_{n=1}^{\infty} \left(a_n - \frac{1}{3}\right)$.

38. Rhombus $ARML$ has its vertices on the graph of $y = 2\lfloor x \rfloor - x$. Given that the area of $ARML$ is 8, compute the least upper bound for $\tan A$.

39. Let N be the number of polynomials such that its coefficients belong to the set $\{1, 2, \ldots, 2018\}$ and $2(P(x) + 1) = P(x+1) + P(x-1)$. Compute the remainder when N is divided by 1000.

40. Let $d(P, MN)$ denote the smallest distance between point P and line segment MN. Let G be the centroid of $\triangle ABC$ and let X be the point in the plane such that $\frac{d(X,AC)}{AC} = \frac{d(X,AB)}{AB} = \frac{d(X,BC)}{BC}$. Let G_a, G_b, G_c be the feet of the perpendiculars from G to BC, AC and AB respectively. Similarly, let X_a, X_b, X_c be the feet of the perpendiculars from X to BC, AC and AB respectively. If O_1 and O_2 are the circumcenters of $\triangle G_aG_bG_c$ and $\triangle X_aX_bX_c$ respectively, compute the value of $\frac{O_1G + O_2X}{2XG}$.